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# Optical Soliton Solution of Higher Order Nonlinear Schrödinger Equation with Cubic Quintic Nonlinearity in non-Kerr Media

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Article Record: Received Dec 24 2022, Revised Paper Received Jan 21, 2023, Final Acceptance Jan 23, 2023
Available Online

**Abstract**

The pulse propagation beyond ultrashort range in optical communication system within non-Kerr media can be modelled by a higher order nonlinear Schrödinger equation with non-Kerr cubic quintic nonlinearity. This paper deals with various type of travelling wave solution of a higher order cubic-quintic nonlinear Schrodinger equation (HOCQNLSE). Using traveling wave ansatz, the HOCQNLSE is converted to a Liénard equation and the solutions are used to describe the bright and dark optical soliton of HOCQNLSE.

**Keywords:** *Cubic Quintic Nonlinearity, Travelling wave ansatz, Dark and bright opticalsoliton.*

## 1. Introduction

The exact solution of nonlinear evolution equations (NLEE) has a great role to describe various physical phenomena. A large number of NLEEs describe physical phenomena, especially the dynamics of electromagnetic wave in fluid, plasma, optical fiber and so on. The solutions of the NLEEs deriving from the physical model are largely associated to describe phenomenological events in nature. With the development of soliton theory, various direct analytical methods are evolved to solve the NLEEs exactly. Such as: tanh method [1, 2, 3], extended tanh method [4, 5], $(G'/G)$-expansion method [6, 7, 8], modified simple equation method [9, 10, 11, 12], Jacobi elliptic function method [13, 14], first integral method [16, 17, 18], Kudryashov method [19, 20], Generalized Kudryashov method [21] and the list is countless.

In optical fiber, balance between self-phase modulation and anomalous dispersion causes the existence of solitons, a localized wave of permanent form. The solitons in optical fibers are called optical solitons [22, 23], the mathematical description for existence of which is reported in the Ref.[24]. Solitons are used to form a transmission system to increase the performance in optical telecommunications. The experimental observation of which is done by Mollenauer et. al. [25]. Therefore, solitons play an important role in the wave (pulse) propagation through the media of optical fibers. Hence, we are interested to extract this kind of solutions from a proposed model of pulse propagation through optical fibers. Different kind of NLSE type equations has been used to model the propagation of pulse through optical fibers that constitutes optical solitons [26, 27, 28].

## 2. Description of Model:

In this article, we shall consider the cubic-quintic higher order nonlinear Schrodinger equation (NLSE) [29, 30] given by,

$$\frac{\partial u}{\partial z} = i\left(\alpha_1\frac{\partial^2 u}{\partial t^2} + \alpha_2|u|^2u\right) + \alpha_3\frac{\partial^3 u}{\partial t^3} + \left(\alpha_4\frac{\partial}{\partial t}(|u|^2u) + \alpha_5 u\frac{\partial}{\partial t}(|u|^2)\right) + i\alpha_6|u|^4u$$
$$+\left(\alpha_7\frac{\partial}{\partial t}(|u|^4u) + \alpha_8 u\frac{\partial}{\partial t}(|u|^4)\right). \tag{1}$$

This equation was derived to describe the propagation of optical pulse in optical fiber considering the non-Kerr effect. Here $u(z,t)$ is the slowly varying envelope of the electric field, $z$ is the normalized distance along the fiber, and $t$ is the normalized time with the frame of the reference moving along the fiber at the group velocity. The subscripts $z$ and $t$ denote the spatial and temporal partial derivatives, respectively. The coefficients $\alpha_i (i = 1,2,\ldots 5)$ are the real parameters related to group velocity dispersion, self-phase modulation, third order dispersion, self-steepening and self-frequency shift due to stimulated Raman scattering (SRS), while $i$ = 6, 7 and 8 are the quintic non-Kerr terms. When $\alpha_3 = \alpha_7 = \alpha_8 = 0$, the equation is known as Cubic quintic NLSE. If $\alpha_4 = \alpha_5$ occurs in cubic-quintic NLSE, it is known as Dysthe equation. In absence of quintic (non-Kerr) nonlinearity, the equation is known as Hirota equation.

In this article we shall solve the Eq. (1) using the conventional travelling wave ansatz. The transformed equation, a Liénard equation, will again be transformed to our desirable form. Finally, by direct integration technique and using the solution of transformedequation we get our desired solution of Eq. (1).

## 3. Travelling Wave Solution:

Let us consider a travelling wave transform

$$u(z,t) = e^{i\theta}U(z,t) \text{ where } \theta = kz - \omega t \tag{2}$$

Here $k, \omega$ are arbitrary constants and the function $U(z,t)$ is real. The function $\theta$ is called the linear phase shift function. Applying the transformation (2) to Eq. (1) we get the following equations after removing the exponential term:

$$U_z + a_1U_t - (a_4 + a_5)U^2U_t - (a_8 + a_9)U^4U_t - \alpha_3U_{ttt} = 0. \tag{3}$$

$$a_2U_{tt} + a_3U^3 - a_6U + a_7U^5 = 0. \tag{4}$$

where $U_z, U_t, U_{tt}, U_{ttt}$ denotes the partial derivatives of the function $U$ with respect to independent variables $z$ and $t$. The constants $a_1 = -2\alpha_1\omega + 3\alpha_3\omega^2$, $a_2 = \alpha_1 - 3\alpha_3\omega$, $a_3 = \alpha_2 - \alpha_4\omega$, $a_4 = 2\alpha_4 + \alpha_5$, $a_5 = \alpha_4 + \alpha_5$, $a_6 = k + \alpha_1\omega^2 - \alpha_3\omega^3$, $a_7 = \alpha_6 - \alpha_7\omega$, $a_8 = 3\alpha_7 + 2\alpha_8$, $a_9 = 2(\alpha_7 + \alpha_8)$. Now we apply the travelling wave transform $\xi = z - Mt$, $M$ is the travelling wave velocity and the system of PDEs reduces to the system of ODEs:

$$\alpha_3M^3U''' + (1 - Ma_1)U' + M(a_4 + a_5)U^2U' + M(a_8 + a_9)U^4U' = 0. \tag{5}$$

and

$$a_2M^2U'' - a_6U + a_3U^3 + a_7U^5 = 0. \tag{6}$$

Integrating the Eq. (5) once, ignoring the integrating constant, we obtain

$$\alpha_3M^3U'' + (1 - Ma_1)U + \frac{M(a_4 + a_5)}{3}U^3 + \frac{M(a_8 + a_9)}{5}U^5 = 0. \tag{7}$$

The prime refers to the derivative of the function $U$ with respect to the variable $\xi$. Since $U$ satisfy both the equations (6) and (7), it must satisfy a consistency condition

$$\frac{\alpha_3M}{a_2} = \frac{Ma_1 - 1}{a_6} = \frac{M(a_4 + a_5)}{3a_3} = \frac{M(a_8 + a_9)}{5a_7}. \tag{8}$$

Therefore, the ODE of our consideration from (6) and (7) using the constraint (8) is given by:

$$U'' + lU + mU^3 + nU^5 = 0. \tag{9}$$

where $l = -\frac{a_6}{a_2 M^2}$, $m = \frac{a_3}{a_2 M^2}$ and n=$\frac{a_7}{a_2 M^2}$ . Now, the Eq. (9) is a Liénard equation. Multiplying (9) by $2U'$ to both sides and then integrating we have

$$(U')^2 = AU^2 + BU^4 + CU^6. \tag{10}$$

where A= $-l = \frac{a_6}{a_2 M^2}$ , B=$-\frac{m}{2} = -\frac{a_3}{2a_2 M^2}$ , C=$-\frac{n}{3}$=$-\frac{a_7}{3a_2 M^2}$ . Let us consider a transformation given by

$$U(\xi) = \sqrt{F(\xi)} \tag{11}$$

Applying the transformation (11) to Eq. (10) we get the following ODE

$$(F')^2 = AF^2 + BF^3 + CF^4. \tag{12}$$

From equation (12), after an integration we obtain the following:

$$\int \frac{dF}{F\sqrt{A + BF + CF^2}} = \pm(\xi - \xi_0). \tag{13}$$

where $\xi_0$ is an integration constant. Define the discriminant $\Delta = 4AC - B^2$. With respect to signs of $A$, $B$ and $\Delta$, we get the following solutions in $F$ by means of direct integration technique [32].

Case1: (Dark Envelope Soliton) If $\Delta = 0$, $A > 0$ and $B < 0$ then

$$F(\xi) = \frac{2A}{e^{\mp\sqrt{A}(\xi-\xi_0)} - B}. \tag{14}$$

Therefore, the solution is given by

$$u_1(z,t) = \sqrt{F(\xi)}e^{i\theta} = \left(\frac{2A}{e^{\mp\sqrt{A}(\xi-\xi_0)} - B}\right)^{1/2} e^{i\theta}. \tag{15}$$

This solution is a dark envelope soliton. The profile associated with such soliton exhibits dip in a uniform background. Hence it is called a dark envelop soliton.

Case2: (Bright Envelope Soliton) If $\Delta < 0$, $A < 0$ and $B < 0$ then

$$F(\xi) = \frac{2A}{B + \sqrt{-\Delta}\cosh(\sqrt{A}(\xi - \xi_0))}. \tag{16}$$

The next solution is given by

$$u_2(z,t) = \left(\frac{2A}{B + \sqrt{-\Delta}\cosh(\sqrt{A}(\xi - \xi_0))}\right)^{1/2} e^{i\theta}. \tag{17}$$

This solution yields a bright envelope soliton of (1). These solutions above are the possible solution extracted by the means of integration. Now we shall apply the mathematical recipes of the Ref. [33], where the use the transform (11) directly into Eq. (9) to get the following form:

$$2FF'' - (F')^2 + 4lF^2 + 4mF^3 + 4nF^4 = 0. \tag{18}$$

Now assuming Eq. (18) admits solution of the form

$$F(\xi) = \frac{re^{\alpha(\xi-\xi_0)}}{\{1+e^{\alpha(\xi-\xi_0)}\}^2 + se^{\alpha(\xi-\xi_0)}}. \tag{19}$$

where $\alpha$, $r$ and $s$ are constants to be determined and $\xi_0$ is an arbitrary constant. Since we are interested to find a localized solution and Liénard equation has a solution of the form of (19), the assumption to consider the solution of the localized form is justified. Now differentiating (19) with respect to $\xi$ twice, we get

$$F'(\xi) = \frac{r\alpha e^{\alpha(\xi-\xi_0)} - r\alpha e^{3\alpha(\xi-\xi_0)}}{(\{1+e^{\alpha(\xi-\xi_0)}\}^2 + se^{\alpha(\xi-\xi_0)})^2} \tag{20}$$

and

$$F''(\xi) = \frac{r\alpha^2 e^{\alpha(\xi-\xi_0)}\Phi(\xi)}{(\{1+e^{\alpha(\xi-\xi_0)}\}^2 + se^{\alpha(\xi-\xi_0)})^3} \tag{21}$$

where $\Phi(\xi) = 1 - (2+s)e^{\alpha(\xi-\xi_0)} - 6e^{2\alpha(\xi-\xi_0)} - (2+s)e^{3\alpha(\xi-\xi_0)} + e^{4\alpha(\xi-\xi_0)}$. Substituting Eq. (19)-Eq.(21) in Eq.(18) we get equations satisfied by the undetermined coefficients $r$, $s$ and $\alpha$ solving which we get

$$\alpha = \pm\sqrt{-4l},\ r = \pm\sqrt{\frac{192l^2}{3m^2 - 16nl}} \text{ and } s = -2 \pm \frac{m}{4}\sqrt{\frac{192}{3m^2 - 16nl}} \tag{22}$$

Collecting these values of undetermined coefficients and substituting in (19) we get

$$F(\xi) = \frac{\sqrt{\frac{192l^2}{3m^2-16nl}}e^{\sqrt{-4l}(\xi-\xi_0)}}{\{1+e^{\sqrt{-4l}(\xi-\xi_0)}\}^2 + \left(-2 \pm \frac{m}{4}\sqrt{\frac{192}{3m^2-16nl}}\right)e^{\sqrt{-4l}(\xi-\xi_0)}}. \tag{23}$$

Note that the solution is permissible only when $l < 0$ and $n > 0$. Hence, within this region of permissibility, the solution of Eq. (1), a bright soliton in this case is given by

$$u_3(z,t) = \pm\left(\frac{\sqrt{192l^2/(3m^2 - 16ln)}\operatorname{sech}^2(\sqrt{A}(\xi-\xi_0))}{4 + (-2 - 2\sqrt{3}m/3m^2 - 16ln)\operatorname{sech}^2(\sqrt{A}(\xi-\xi_0))}\right)^{1/2} e^{i\theta}. \tag{24}$$

Both the solutions (17) and (24) has a Gaussian shaped profile. If we launch the solutions in a lossless fiber, the pulse will propagate undistorted without change in shape for arbitrarily long distances. On the contrary of dark envelope solitons, this is called bright envelope solitons due to obvious nature.

**4. Conclusion:**

The nonlinear Schrodinger equation is a powerful mathematical model which has found wide application in the nonlinear description of many different physical systems, including water waves, Bose-Einstein condensates, and plasmas. One of the most important applications of the equation, however, has been in the description of pulse propagation in single-mode optical waveguides. In this form the equation predicts many well-characterized phenomena including soliton propagation, modulation instability, self-steepening, and parabolic pulse propagation for various appropriate combinations of the nonlinearity and dispersion parameters. In this paper, we have investigated a HOCQNLSE modelling the propagation of ultrashort (femtosecond) optical pulses in nonlinear optical fibers. The model used combines cubic and quintic nonlinearities, as well as the self-steepening and self-frequency shift effects. Thus, the

solitary wave solutions for the HOCQNLSE equation with non-Kerr term can be applied practically to the systems to describe the propagation of dark optical pulses beyond femtosecond embedded with finite-width background bright pulses because here ideally group velocity dispersion and third order dispersion approaches zero value i.e. the solitary wave cannot be destroyed. Thus, the solitary envelopes, could find applications in telecommunication because the dark pulses are less prone to disperse than regular bright pulses, but also the HOCQNLSE with non-Kerr terms may be the new theoretical model equation for experimental designing of Photonic Crystal Fiber to achieve highly efficient dark soliton pulse propagation and to study the laser pulse compression dynamics to achieve significant enhancement of the spectral width.
The Lienard equation method is used to construct families of bright and dark solutions for the nonlinear dynamical model. Conditions for the existence of propagating envelope solutions have also been reported. The solutions may be important to study some physical phenomena in the similar physical context and the mathematical technique again shows the importance of direct methods in handling the NLEEs arising from nature.

## *References*

[1] Malfiet, W. & Hereman, W. (1996). The tanh method: I. Exact solutions of nonlinear evolution and wave equations. *Physica Scripta*, 54, 563-568.
[2] Wazwaz, A. M. (2004). The tanh method for traveling wave solutions of nonlinear equations. *Applied Mathematics and Computation*, 54, 713-723.
[3] Wazwaz, A. M. (2005). The tanh method for generalized forms of nonlinear heat conduction and Burgers-Fisher equations. *Applied Mathematics and Computation*, 169, 321-338.
[4] Fan, E. (2000). Extended tanh-function method and its applications to nonlinear equations. *Physics Letter A*, 277, 212-218.
[5] Wazwaz, A. M. (2007). The extended tanh method for abundant solitary wave solutions of nonlinear wave equations. *Applied Mathematics and Computation*, 187, 1131-1142.
[6] Wang, M., Li, X., & Zhang, J. (2008). The (G`/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. *Physics Letter A*, 372, 417-423.
[7] Bekir, A. (2008). Application of the (G`/G)-expansion method for nonlinear evolution equations. *Physics Letter A*, 372, 3400-3404.
[8] Kudryashov, N. A. (2010). A note on the (G`/G)-expansion method. *Applied Mathematics and Computation*, 217, 1755-1758.
[9] Jawad, A. J. M., Petrovic, M. D., & Biswas, A. (2010). Modified simple equation method for nonlinear evolution equations. *Applied Mathematics and Computation*, 217, 869-877.
[10] Arnous, A. M., Seadawy, A. R., Alqahtani, R.T., & Biswas, A. (2017). Optical solitons with complex Ginzburg-Landau equation by modified simple equation method. *Optik*, 144, 475-480.
[11] Biswas, A., Yildirim, Y., Yasar, E., Triki, H., Alshomrani, A. S., Ullah, M. Z., Zhou, Q., Moshokoa, S. P., & Belic, M. (2018). Optical soliton perturbation for complex Ginzburg-Landau equation with modified simple equation method, *Optik*, 158, 399-415.
[12] Biswas, A., Yildirim, Y., Yasar, E., Zhou, Q., Moshokoa, S. P., & Belic, M. (2018). Optical solitons for Lakshmanan-Porsezian-Daniel model by modified simple equation method. *Optik*, 160, 24-32.
[13] Parkes, E. J., Duffy, B. R., & Abbott, P. C. (2002). The Jacobi elliptic-function method for finding periodic-wave solutions to nonlinear evolution equations. *Physics Letter A*, 295, 280-286.

[14] Fan, E., & Zhang, J. (2002). Applications of the Jacobi elliptic function method to special type nonlinear equations. *Physics Letter A*, 305, 383-392.
[15] Feng, Z. (2002). The first-integral method to study the Burgers-Korteweg de-Vries equation. *Journal of Physics A: Mathematical and General*, 35, 343-349.
[16] Taghizadeh, N., Mirzazadeh, M., & Farahrooz, F. (2011). Exact solutons of the nonlinear Schrodinger equation by first integral method. *Journal of Mathematical Analysis and Applications*, 374, 549-553.
[17] Abbasbandy, S., & Shirzadi, A. (2010). The first integral method for modified Benjamin-Bona-Mahony equation. *Communications of Nonlinear Sciences and Numerical Simulation*, 15, 1759-1764.
[18] Majee, D. N. (2022). Grey optical dips in KMN model by first integral method. *International Journal of Research and Analytical Reviews (IJRAR)*, 9, 657-660.
[19] Kudryashov, N. A. (1990). Exact solutions of the generalized Kuramoto-Sivashinsky equation. *Physics Letter A*, 147, 287-291.
[20] Gonzalez-Gaxoila, O., Leon-Ramirez, A., & Chacon-Acosta, G. (2022). Application of the Kudryashov Method for Finding Exact Solutions of the Schamel-Kawahara Equation. *Russian Journal of Nonlinear Dynamics,* 18, 203-215.
[21] Kaplan, M., & Bekir, A., & Akbulut, A. (2016). *Nonlinear Dynamics*, 85, 2843-2850.
[22] Hasegawa, A., & Tappert, F. (1973). Transmission of Stationary Nonlinear Optical Physics in Dispersive Dielectric Fibers I: Anomalous Dispersion. *Applied Physics Letters*, 23, 142-144.
[23] Hasegawa, A., & Tappert, F. (1973). Transmission of Stationary Nonlinear Optical Physics in Dispersive Dielectric Fibers II: Anomalous Dispersion. *Applied Physics Letters*, 23, 171-172.
[24] Eilbeck, J. C., Gibbon, J. D., Caudrey, P. J., & Bullough, R. K. (1973). Solitons in nonlinear optics 1: more accurate description of 2pi pulse in self-induced transparency. *Journal of Physics A: Mathematical and General*, 6, 1337-1347.
[25] Mollenauer, L. F., Stolen, R. H., & Gordon, J. P. (1980). "Experimental Observation of Picosecond Pulse Narrowing and Solitons in Optical Fiber". *Physical Review Letters*, 45, 1095-1097.
[26] Isah, M. A., & Yokus, A. (2022). The investigation of several soliton solutions to the complex Ginzburg-Landau model with Kerr law nonlinearity. *Mathematical Modelling and Numerical Simulation with Applications*, 2, 147-163.
[27] Seadawy, A. R., Rizvi, S. T. R., Akram, U., & Naqvi, S. K. (2022). Optical and analytical soliton solutions to higher order non-Kerr nonlinear Schrodinger dynamical model. *Journal of Geometry and Physics*, 179, 104616.
[28] Kudryashov, N. A., & Biswas, A. (2022). Optical solitons of nonlinear Schrodinger's equation with arbitrary dual-power law parameters. *Optik*, 252, 168497.
[29] Choudhuri, A., & Porsezian, K. (2012). Dark-in-the-Bright solitary wave solution of higherorder nonlinear Schrodinger equation with non-Kerr terms. *Optics Communications*, 285, 364-367.
[30] Choudhuri, A., & Porsezian, K. (2013). Higher-order nonlinear Schrodinger equation with derivative non-Kerr nonlinear terms: A model for sub-10-fs-pulse propagation. *Physical Review A*, 88, 033808(1)-033808(8).
[31] Triki, H., Zhou, Q., Moshokoa, S. P., Ullah, M. Z., Biswas, A., & Belic, M. (2018). Gray and black optical solitons with quintic nonlinearity. *Optik*, 154, 354-359.
[32] Gradshteyn, I. S., Ryzhik, I. M. (2007). Table of Integrals, Series, and Products, Academic Press, New York.
[33] Kong, D. (1995). Explicit exact solutions for the Lienard equation and its applications, *Physics Letters A*, 196, 301-306.