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On Some Prime Fuzzy Ideals of Semi rings

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Abstract

In this paper we introduce and study the concepts of weakly Prime fuzzy ideal, prime fuzzy left ideal, completely Prime, weakly completely Prime fuzzy ideal and quasi-prime, weakly quasi-prime fuzzy left ideal of semi rings. Also, we have shown for a commutative semi ring the notions of prime, quasi-prime and weakly quasi-prime fuzzy left ideals are equivalent.

Keywords: *Weakly prime fuzzy ideal, prime fuzzy left ideal, completely prime and weakly completely prime fuzzy ideals, quasi-prime fuzzy left ideal, weakly quasi-prime fuzzy left ideal.*

1. Introduction

In the past several years there has been remarkable growth in the interest of fuzzy set theory and its applications. The concept of fuzzy set, introduced by Lotfi Zadeh, provides a frame work for generalizing several basic notions of algebra. Xie (2001) introduced and characterized the notions of prime, quasi-prime, weakly quasi-prime fuzzy left ideals of semigroups. Also Tang (2012) introduced and studied the notions of weakly prime and weakly quasi-prime fuzzy left ideals in ordered semigroups. In this paper we introduce and characterize the notions of weakly Prime fuzzy ideal, prime fuzzy left ideal, completely Prime, weakly completely Prime fuzzy ideal and quasi-prime, weakly quasi-prime fuzzy left ideal of semi rings.

2. Preliminaries

Definition 2.1 A nonempty set S is said to form a semi ring with respect to two binary compositions, addition (+) and multiplication(.) defined on it, if the following conditions are satisfied.
(1) (S, +) is a commutative semigroup with zero,
(2) (S,.) is a semigroup,
(3) for any three elements $a, b, c \in S$
the left distributive law $a.(b + c) = a.b + a.c$ and
the right distributive law $(b + c).a = b.a + c.a$ both hold and
(4) $s.0 = 0.s = 0$ for all $s \in S$.

Definition 2.2 A nonempty subset I of a semi ring S is called a left (right) ideal of S if $a, b \in I$ implies $a + b \in I$ and (ii) $a \in I$, $s \in S$ implies $s.a \in I$ ($a.s \in I$).
A nonempty subset I of a semi ring S is an ideal if it is a left ideal as well as a right ideal of S.

Definition 2.3 An ideal I of a semi ring S is said to be a prime ideal of S if A and B are two ideals of S such that $AB \subseteq I$ then either $A \subseteq I$ or $B \subseteq I$.

Definition 2.4 An ideal I of a semi ring S is said to be a completely prime ideal of S, if $xy \in S$, $x, y \in S$, then $x \in I$ or $y \in I$.

Definition 2.5 A left ideal L of a semi ring S is said to be a quasi-prime left ideal of S, if for any two left ideals L_1 and L_2 of S, $L_1L_2 \subseteq L$ implies $L_1 \subseteq L$ or $L_2 \subseteq L$.

Definition 2.6 A left ideal L of a semi ring S is said to be a weakly quasi-prime left ideal of S, if L_1 and L_2 are two left ideals of S with $L \subseteq L_1$, $L \subseteq L_2$ and $L_1L_2 \subseteq L$, then $L_1 \subseteq L$ or $L_2 \subseteq L$.

Definition 2.7 [Biswas, B. K. & Dutta, T. K. (1997)] Let μ be a nonempty fuzzy subset of a semiring S (i.e. $\mu(x) \neq 0$ for some $x \in S$). Then μ is called a fuzzy left [fuzzy right] ideal of S if
(1) $\mu(x + y) \geq \min[\mu(x), \mu(y)]$ and
(2) $\mu(xy) \geq \mu(y)$ [resp. $\mu(xy) \geq \mu(x)$], $\forall x, y \in S$.
A fuzzy ideal of a semi ring S is a nonempty fuzzy subset of S which is a fuzzy left ideal as well as a fuzzy right ideal of S.

Definition 2.8 [Biswas, B. K. & Dutta, T. K. (1997)] A fuzzy ideal μ of a semi ring S is said to be prime fuzzy ideal if μ is not a constant function and for any two fuzzy ideals μ_1 and μ_2 of S, $\mu_1 o \mu_2 \subseteq \mu$ implies either $\mu_1 \subseteq \mu$ or $\mu_2 \subseteq \mu$.

Definition 2.9 [Biswas, B. K. & Dutta, T. K. (1997)] If μ_1 and μ_2 are two fuzzy subsets of a semi ring S then their union $\mu_1 \cup \mu_2$ and intersection $\mu_1 \cap \mu_2$ are respectively defined by $(\mu_1 \cup \mu_2)(x) = \max[\mu_1(x), \mu_2(x)]$ and $(\mu_1 \cap \mu_2)(x) = \min[\mu_1(x), \mu_2(x)]$ for all $x \in S$.

Definition 2.10 Let S be a semiring, $x \in S$, $r \in (0, 1]$. We define x_r as follows:

$$x_r(s) = r \text{ if } s = x$$
$$= 0 \text{ otherwise}$$

x_r is called a fuzzy point of S.

Definition 2.11 (Biswas, B. K. & Dutta, T. K. (1997) Let S be a semi ring and μ_1, μ_2 be two fuzzy subsets of S. Then composition of μ_1 and μ_2, denoted by $\mu_1 o \mu_2$ and is defined by

$$\mu_1 \text{ o } \mu_2(x) = \sup_{x=uv} [\min[\mu_1(u), \mu_2(v)]],$$
$$= 0 \text{ if } x \text{ is not expressible as } x = uv \text{ for any } u, v \in S.$$

Proposition 2.12, Let μ_1, μ_2 and μ_3 be fuzzy subsets of a semi ring S. Then
$\mu_1 \text{ o } (\mu_2 \cup \mu_3) = \mu_1 \text{ o } \mu_2 \cup \mu_1 \text{ o } \mu_3$.

Proof: Let $x \in S$. If x is not expressible as product of two elements of S, then
$(\mu_1 \text{ o } (\mu_2 \cup \mu_3))(x) = 0 = (\mu_1 \text{ o } \mu_2 \cup \mu_1 \text{ o } \mu_3)(x)$
If x is expressible as product of two elements of S, then

$$(\mu_1 \text{ o } (\mu_2 \cup \mu_3))(x) = \sup_{x=uv} \min[\mu_1(u), \max\{\mu_2(v), \mu_3(v)\}]$$
$$= \max\{\sup_{x=uv} \min[\mu_1(u), \mu_2(v)], \sup_{x=uv} \min[\mu_1(u), \mu_3(v)]\}$$
$$= \max\{\mu_1 \text{ o } \mu_2(x), \mu_1 \text{ o } \mu_3(x)\}$$
$$= (\mu_1 \text{ o } \mu_2 \cup \mu_1 \text{ o } \mu_3)(x)$$

Therefore $(\mu_1 \text{ o } (\mu_2 \cup \mu_3))(x) = (\mu_1 \text{ o } \mu_2 \cup \mu_1 \text{ o } \mu_3)(x)$, $\forall x \in S$.
Hence $\mu_1 \text{ o } (\mu_2 \cup \mu_3) = \mu_1 \text{ o } \mu_2 \cup \mu_1 \text{ o } \mu_3$.

Proposition 2.13 Let x_r be a fuzzy point of S. Then the following conditions hold:

(i) The fuzzy left ideal of S generated by x_r is
$< x_r >_L (s) = r$ if $s \in < x >_L$
$= 0$ otherwise $\forall s \in S$.
where $< x >_L$ is the left ideal of S generated by x.
(ii) The fuzzy ideal of S generated by x_r is
$< x_r > (s) = r$ if $s \in < x >$
$= 0$ otherwise $\forall s \in S$.
where $< x >$ is the ideal of S generated by x.

Proof: (i) Let H be a fuzzy subset of S such that
$H(s) = r$ if $s \in < x >_L$
$= 0$ otherwise $\forall s \in S$.
We now prove that H is the smallest fuzzy left ideal of S containing x_r.
Clearly $x_r \in H$. Let $a, b \in S$. Then $\min\{H(a),H(b)\} = 0$ or r.
If $\min\{H(a),H(b)\} = 0$, then $H(a + b) \geq \min\{H(a),H(b)\} = 0$.
If $\min\{H(a),H(b)\} = r$, then $H(a) = r = H(b)$ i.e. $a, b \in < x >_L$. So $a + b \in < x >_L$. Thus $H(a + b) = r$.
Therefore $H(a + b) \geq \min\{H(a),H(b)\}$, $\forall a, b \in S$.
Now $H(b) = 0$ or r. If $H(b) = 0$, then $H(ab) \geq H(b)$. If $H(b) = r$, then $b \in < x >_L$. Since $< x >_L$ is a left ideal of S, $ab \in < x >_L$ i.e. $H(ab) = r$. So $H(ab) \geq H(b)$
Therefore, $H(ab) \geq H(b)$, $\forall a, b \in S$.
Hence H is a fuzzy left ideal of S containing x_r.
Let P be any fuzzy left ideal of S containing x_r. If possible, let H be not subset of P. So there exists $y \in S$ such that $P(y) < H(y)$. So $H(y) = r$ i.e. $y \in < x >_L$. Therefore, $y = nx$ or $y = sx$, where n is a non-negative integer and $s \in S$. Since P is a fuzzy left ideal of S containing xr, in both cases $P(y) \geq P(x) \geq r$, a contradiction. Thus H is the smallest fuzzy left ideal of S containing x_r. Hence proved.

(ii) The proof is similar to the proof of (i).

Proposition 2.14 Let S be a semiring and x_r, y_t be any two fuzzy points of S. Then the following statements hold:
(i) $(S \text{ o } x_r \text{ o } S)(s) = r$ if $s \in SxS$.
$= 0$ otherwise.
$(x_r \text{ o } S)(s) = r$ if $s \in xS$.
$= 0$ otherwise.
$(S \text{ o } x_r)(s) = r$ if $s \in Sx$.
$= 0$ otherwise.
(ii) $x_r \text{ o } y_t = (xy)_{\min\{r,t\}}$.
(iii) $< x_r >_L = x_r \cup S \text{ o } x_r$, $< x_r > = x_r \cup x_r \text{ o } S \cup S \text{ o } x_r \cup S \text{ o } x_r \text{ o } S$.
(iv) $< x_r >^3 \subseteq S \text{ o } x_r \text{ o } S$.

Proof: (i) If $s \in SxS$, then $s = s_1xs_2$, where $s_1, s_2 \in S$. Now $C_S(s_1) = 1$, $x_r(x) = r$ and $C_S(s_2) = 1$.
Therefore $\min\{C_S(s_1), x_r(x), C_S(s_2)\} = r$. Thus $(S \text{ o } x_r \text{ o } S)(s) = \sup\limits_{s=abc} \min\{C_S(a), x_r(b), C_S(c)\} = r$.

If s does not belong to SxS, then at least one image of C_S or x_r is 0. So $(S \text{ o } x_r \text{ o } S)(s) = 0$.
Hence $(S \text{ o } x_r \text{ o } S)(s) = r$ if $s \in SxS$.
$= 0$ otherwise.
The proof of the other two parts are similar.

(ii) Let $a \in S$. If $a = xy$, then $(x_r \text{ o } y_t)(a) = \sup\limits_{a=cd} \min\{x_r(c), y_t(d)\} = \min\{x_r(x), y_t(y)\} = \min\{r, t\} =(xy)_{\min\{r,t\}}(a)$.

If $a \neq xy$, then $(x_r \text{ o } y_t)(a) = 0 = (xy)\min\{r,t\}$. Thus $x_r \text{ o } y_t = (xy)_{\min\{r,t\}}$.

(iii) Follows from (i) and 2.13.

(iv) Since $< x_r >= x_r \cup x_r \text{ o } S \cup S \text{ o } x_r \cup S \text{ o } x_r \text{ o } S$ and $< x_r > \subseteq S$,
$< x_r >^3 \subseteq S \text{ o } (x_r \cup x_r \text{ o } S \cup S \text{ o } x_r \cup S \text{ o } x_r \text{ o } S) \text{ o } S$
$\subseteq (S \text{ o } x_r \cup S \text{ o } x_r \text{ o } S) \text{ o } S$ (by Proposition 2.12)
$\subseteq S \text{ o } x_r \text{ o } S$.

Definition 2.15 Let H be a subset of a semiring S and $t \in [0, 1]$. We define fuzzy subset tC_H as follws:
$tC_H(s) = t$ if $s \in H$
$= 0$ otherwise $\forall s \in S$.

Proposition 2.16 Let S be a semiring and H, K be two subsets of S. Then for any $t \in (0, 1]$ the following statements hold:
(i) $tC_H \text{ o } tC_K = tC_{HK}$.
(ii) $S \text{ o } tC_H = tC_{SH}$.
(iii) If I is an ideal (right ideal, left ideal) of S, then tC_I is a fuzzy ideal (resp. fuzzy right ideal, fuzzy left ideal) of S.

Proof: (i) Let $x \in S$. Now $tC_H \text{ o } tC_K(x) = t$ or 0 and $tC_{HK}(x) = t$ or 0.
$tC_H \text{ o } tC_K(x) = t$
$\Leftrightarrow \sup_{x=ab} \min\{tC_H(a), tC_K(b)\} = t$
$\Leftrightarrow x = ab$, $tC_H(a) = t$ and $tC_K(b) = t$
$\Leftrightarrow x = ab$, $a \in H$ and $b \in K$
$\Leftrightarrow x \in HK \Leftrightarrow tC_{HK}(x) = t$.
Thus $tC_H \text{ o } tC_K = tC_{HK}$.

(ii) Similar as (i)

(iii) Now $tC_I(x) = t$ or 0, $\forall x \in S$.
Let $x, y \in S$. If $\min\{tC_I(x), tC_I(y)\} = 0$, then $tC_I(x + y) \geq \min\{tC_I(x), tC_I(y)\}$. If $\min\{tC_I(x), tC_I(y)\} = t$, then $tC_I(x) = t$ and $tC_I(y) = t$ i.e. $x, y \in I$. Since I is an ideal, $x + y \in I$. So $tC_I(x + y) = t$.
Thus $tC_I(x + y) \geq \min\{tC_I(x), tC_I(y)\}$, $\forall x, y \in S$.
If $\max\{tC_I(x), tC_I(y)\} = 0$, then $tC_I(xy) \geq \max\{tC_I(x), tC_I(y)\}$.
If $\max\{tC_I(x), tC_I(y)\} = t$, then $tC_I(x) = t$ or $tC_I(y) = t$ i.e. $x \in I$ or $y \in I$. Since I is an ideal, $xy \in I$. So $tC_I(xy) = t$.
Therefore, $tC_I(xy) \geq \max\{tC_I(x), tC_I(y)\}$, $\forall x, y \in S$.
Hence tC_I is a fuzzy ideal of S.
Similarly, if I is a right (left) ideal of S, then C_I is a fuzzy right (left) ideal of S.

3. Weakly Prime Fuzzy Ideals

Definition 3.1 A fuzzy ideal μ of a semiring S is called a weakly prime fuzzy ideal of S if for all ideals H and K of S and for all $t \in (0, 1]$, $tC_H \text{ o } tC_K \subseteq \mu$ implies $tC_H \subseteq \mu$ or $tC_K \subseteq \mu$.

Theorem 3.2 Let μ be a fuzzy ideal of a semiring S. Then μ is a weakly prime fuzzy ideal of S if and only if
$\mu_t\,(t > 0)$ is a prime ideal of S for $\mu_t \neq \varphi$.

Proof: Let μ be a weakly prime fuzzy ideal of S and $HK \subseteq \mu_t$, where H, K are two ideals of S and $t \in (0, 1]$. So $tC_{HK} \subseteq \mu$ i.e. $tC_H \text{ o } tC_K \subseteq \mu$ (by Proposition 2.16). Since μ is a weakly prime fuzzy ideal of S, $tC_H \subseteq \mu$ or $tC_K \subseteq \mu$. Therefore, $H \subseteq \mu_t$ or $K \subseteq \mu_t$. Hence μ_t is a prime ideal of S.

Again let μ_t ($t \in (0, 1]$) be a prime ideal of S and H, K be two ideals of S such that tC_H o $tC_K \subseteq \mu$. So $tC_{HK} \subseteq \mu$ i.e. $HK \subseteq \mu_t$. Since μt is a prime ideal of S, $H \subseteq \mu_t$ or $K \subseteq \mu_t$ i.e. $tC_H \subseteq \mu$ or $tC_K \subseteq \mu$. Therefore, μ is a weakly prime fuzzy ideal of S.

Theorem 3.3 Let μ be a fuzzy ideal of a semiring S. Then the following statements are equivalent:
(i) μ is a weakly prime fuzzy ideal of S.
(ii) x_r o S o $y_r \subseteq \mu \Rightarrow x_r \in \mu$ or $yr \in \mu$, where $x, y \in S$ and $r \in (0, 1]$.
(iii) $< x_r >$ o $< y_r > \subseteq \mu \Rightarrow x_r \in \mu$ or $y_r \in \mu$, where $x, y \in S$ and $r \in (0, 1]$.
(iv) tC_H o $tC_K \subseteq \mu$ implies that $tC_H \subseteq \mu$ or $tC_K \subseteq \mu$, where H, K are right ideals of S and $t \in (0, 1]$.
(v) tC_H o $tC_K \subseteq \mu$ implies that $tC_H \subseteq \mu$ or $tC_K \subseteq \mu$, where H, K are left ideals of S and $t \in (0, 1]$.
(vi) tC_H o $tC_K \subseteq \mu$ implies that $tC_H \subseteq \mu$ or $tC_K \subseteq \mu$, where H is a right ideal and K is a left ideal of S, $t \in (0, 1]$.

Proof: (i) ⇒ (ii) Let μ be a weakly prime fuzzy ideal of S and x_r o S o $y_r \subseteq \mu$, where $x, y \in S$ and $r \in (0, 1]$. So rC_{SxS} o rC_{SyS} = (S o x_r o S) o (S o y_r o S) ⊆ S o (x_r o S o y_r) o S ⊆ S o μ o S ⊆ μ. Since μ is a fuzzy weakly prime ideal of S, $rC_{SxS} \subseteq \mu$ or $rC_{SyS} \subseteq \mu$ i.e. $< x_r >^3 \subseteq \mu$ or $< y_r >^3 \subseteq \mu$. Therefore $< x_r > = rC_{<x>} \subseteq \mu$ or $< y_r > = rC_{<y>} \subseteq \mu$ which implies that $x_r \in \mu$ or $y_r \in \mu$.
(ii) ⇒ (iii) Let $< x_r >$ o $< y_r > \subseteq \mu$, where $x, y \in S$ and $r \in (0, 1]$. Now x_r o S o $y_r \subseteq < x_r >$ o S o $< y_r > \subseteq < x_r >$ o $< y_r > \subseteq \mu$. By (ii), $x_r \in \mu$ or $y_r \in \mu$.
(iii) ⇒ (iv) Suppose tC_H o $tC_K \subseteq \mu$, where H, K are right ideals of S and $t \in (0, 1]$. If possible, let tC_H is not subset of μ, so there exists $x \in H$ such that x_t does not belong to μ. For any $y \in K$,
$< x_t >$ o $< y_t > = tC_{<x>}$ o $tC_{<y>} = tC_{<x><y>}$
$\subseteq tC_{(H \cup SH)(K \cup SK)}$
$\subseteq tC_{(HK) \cup (SHK)}$
$= (tC_H$ o $tC_K) \cup (S$ o tC_H o $tC_K) \subseteq \mu$
Since x_t does not belong to μ, by (iii), $y_t \in \mu$. Therefore, $tC_K = \cup_{y \in K} y_t \subseteq \mu$.
(iv) ⇒ (i) is obvious.
(iii) ⇒ (v) Suppose tC_H o $tC_K \subseteq \mu$, where H, K are left ideals of S and $t \in (0, 1]$. If possible, let tC_H be not subset of μ, so there exists $x \in H$ such that x_t does not belong to μ. For any $y \in K$,
$< x_t >$ o $< y_t > = tC_{<x>}$ o $tC_{<y>} = tC_{<x><y>}$
$\subseteq tC_{(H \cup HS)(K \cup KS)}$
$\subseteq t_{C(HK) \cup (HKS)}$
$= (tC_H$ o $tC_K) \cup (tC_H$ o tC_K o $S) \subseteq \mu$
Since x_t does not belong to μ, by (iii), $y_t \in \mu$. Therefore, $tC_K = \cup_{y \in K} y_t \subseteq \mu$.
(v) ⇒ (i) is obvious.
(iii) ⇒ (vi) Let H be a right ideal and K be a left ideal of S such that tC_H o $tC_K \subseteq \mu$, where $t \in (0, 1]$. If possible, let tC_H be not subset of μ, so there exists $x \in H$ such that x_t does not belong to μ. Let $y \in K$ be arbitrary. Then $< x_t >$ o $< y_t > = tC_{<x>}$ o $tC_{<y>} = tC_{<x><y>}$
$\subseteq tC_{(H \cup SH)(K \cup KS)} = tC_{(HK) \cup (HKS) \cup (SHK) \cup (SHKS)}$
$= (tC_H$ o $tC_K) \cup (tC_H$ o tC_K o $S) \cup (S$ o tC_H o $tC_K) \cup (S$ o tC_H o tC_K o $S) \subseteq \mu$
By hypothesis either $x_t \in \mu$ or $y_t \in \mu$. Since x_t does not belong to μ, $y_t \in \mu$. Therefore $tC_K = \cup_{y \in K} y_t \subseteq \mu$. Thus either $tC_H \subseteq \mu$ or $tC_K \subseteq \mu$. Hence proved.
(vi) ⇒ (i) is obvious.

4. Prime Fuzzy Left Ideals

Definition 4.1 A fuzzy left ideal μ of a semi ring S is called a prime fuzzy left ideal of S if for any two fuzzy ideals μ_1 and μ_2 of S with μ_1 o $\mu_2 \subseteq \mu$ implies $\mu_1 \subseteq \mu$ or $\mu_2 \subseteq \mu$.

Theorem 4.2 A fuzzy left ideal μ of S is prime if and only if for any two fuzzy points x_r, y_t of S ($r, t \in (0, 1]$) x_r o S o y_t o $S \subseteq \mu$ implies that either $x_r \in \mu$ or $y_t \in \mu$.

Proof: Suppose μ is a prime fuzzy left ideal of S and for any two fuzzy points x_r, y_t (r, t $\in$ (0, 1]) of S, x_r o S o y_t o S $\subseteq \mu$. Then (S o x_r o S) o (S o y_t o S) $\subseteq$ S o (x_r o S o y_t o S) $\subseteq$ S o $\mu \subseteq \mu$.
As S o x_r o S and S o y_t o S are two fuzzy ideals of S and μ is prime, S o x_r o S $\subseteq \mu$ or S o y_t o S $\subseteq \mu$
i.e. $< x_r >^3 \subseteq \mu$ or $< y_t >^3 \subseteq \mu$
i.e. $< x_r > \subseteq \mu$ or $< y_t > \subseteq \mu$
i.e. $x_r \in \mu$ or $y_t \in \mu$.

Conversely, let μ_1, μ_2 be two fuzzy ideals of S such that μ_1 o $\mu_2 \subseteq \mu$.
If possiple, let μ_1 and μ_2 be not subsets of μ. So there exist x, y $\in$ S such that $\mu(x) < \mu_1(x)$ and $\mu(y) < \mu_2(y)$.
Let $\mu_1(x)$ = r and $\mu_2(y)$ = t. Therefore, $x_r \in \mu_1$, $y_t \in \mu_2$ but x_r and y_t do not belong to μ. Thus x_r o S o y_t o S $\subseteq \mu_1$ o $\mu_2 \subseteq \mu$. By hypothesis, $x_r \in \mu$ or $y_t \in \mu$, a contradiction. Hence μ_1 o $\mu_2 \subseteq \mu$ implies that either $\mu_1 \subseteq \mu$ or $\mu_2 \subseteq \mu$. Thus μ is a prime fuzzy left ideal of S.

Theorem 4.3 A left ideal L of S is prime if and only if C_L is a prime fuzzy left ideal of S.

Proof: Let L be a prime left ideal of S. Clearly C_L is a fuzzy left ideal of S. Let μ_1 and μ_2 be two fuzzy ideals of S such that μ_1 o $\mu_2 \subseteq C_L$. If possible, let μ_1 and μ_2 be not subsets of C_L. So there exist x, y $\in$ S such that $\mu_1(x) > C_L(x)$ and $\mu_2(y) > C_L(y)$. Thus $C_L(x) = C_L(y) = 0$ and $\mu_1(x) > 0$, $\mu_2(y) > 0$. So x, y do not belong to L and $\min\{\mu_1(x), \mu_2(y)\} > 0$. We now show that there exist s_1, $s_2 \in$ S such that xs_1ys_2 does not belong to L. If not then xSyS $\subseteq$ L. Thus (SxS)(SyS) $\subseteq$ L. Since L is prime left ideal of S, SxS $\subseteq$ L or SyS $\subseteq$ L
i.e. $< x >^3 \subseteq$ L or $< y >^3 \subseteq$ L
i.e. $< x > \subseteq$ L or $< y > \subseteq$ L
i.e. x $\in$ L or y $\in$ L, a contradiction.
Now $\mu_1 o \mu_2(xs_1ys_2) = \sup_{xs_1ys_2=ab} \min\{\mu_1(a), \mu_2(b)\} \geq \min\{\mu_1(xs_1), \mu_2(ys_2\} \geq \min\{\mu_1(x), \mu_2(y)\} > 0$.
Also $C_L(xs_1ys_2) = 0$, which is a contradiction as μ_1 o $\mu_2 \subseteq C_L$. Thus C_L is a prime fuzzy left ideal of S.

Conversely, suppose C_L is a prime fuzzy left ideal of S and H, K are two ideals of S such that HK $\subseteq$ L. So C_H o $C_K = C_{HK} \subseteq C_L$. Since C_L is a prime fuzzy left ideal of S, $C_H \subseteq C_L$ or $C_K \subseteq C_L$. Therefore, H $\subseteq$ L or K $\subseteq$ L. Thus L is a prime left ideal of S.

5. Completely Prime and Weakly Completely Prime Fuzzy Ideals

Definition 5.1 A fuzzy ideal μ of S is called a completely prime fuzzy ideal of S if for any two fuzzy points x_r, y_t of S (r, t $\in$ (0, 1]), x_r o $y_t \in \mu$ implies that $x_r \in \mu$ or $y_t \in \mu$.

Definition 5.2 A fuzzy ideal μ of S is called a weakly completely prime fuzzy ideal of S if for any fuzzy points x_r, y_r of S (r $\in$ (0, 1]), x_r o $y_r \in \mu$ implies that $x_r \in \mu$ or $y_r \in \mu$.

Theorem 5.3 A fuzzy ideal μ of S is completely prime if and only if for any fuzzy subsets μ_1 and μ_2 of S, μ_1 o $\mu_2 \subseteq \mu$ implies that $\mu_1 \subseteq \mu$ or $\mu_2 \subseteq \mu$.

Proof: Suppose μ is a completely prime fuzzy ideal of S and μ_1 o $\mu_2 \subseteq \mu$. Let μ_1 be not subset of μ, then there exists $x_r \in \mu_1$ (r $\in$ (0, 1]) such that x_r does not belong to μ. Let $y_t \in \mu_2$ (t $\in$ (0, 1]). So x_r o $y_t \subseteq \mu_1$ o $\mu_2 \subseteq \mu$. Since μ is a completely prime fuzzy ideal of S, $x_r \in \mu$ or $y_t \in \mu$. As x_r does not belong to μ, $y_t \in \mu$.
Thus $\mu_2 = \bigcup_{y_t \in \mu_2} y_t \subseteq \mu$ i.e. $\mu_1 o \mu_2 \subseteq \mu$ implies that $\mu_1 \subseteq \mu$ or $\mu_2 \subseteq \mu$.
The converse part is obvious.

Theorem 5.4 A fuzzy ideal μ of a semiring S is weakly completely prime if and only if $\mu(xy) = \max\{\mu(x), \mu(y)\}$, $\forall$x, y $\in$ S.

Proof: Suppose μ is a weakly completely prime fuzzy ideal of S. Since μ is a fuzzy ideal of S, $\mu(xy) \geq \max\{\mu(x), \mu(y)\}$, $\forall x, y \in S$.......(1)
Let $\mu(xy) = r$. So $(xy)_r \in \mu$ i.e. $x_r \circ y_r \in \mu$ (by Proposition 2.14(ii)). As μ is a weakly completely prime fuzzy ideal of S, $x_r \in \mu$ or $y_r \in \mu$ i.e. $r \leq \mu(x)$ or $r \leq \mu(y)$. So $r \leq \max\{\mu(x), \mu(y)\}$.
Therefore, $\mu(xy) \leq \max\{\mu(x), \mu(y)\}$, $\forall x, y \in S$.......(2)
From (1) and (2) we get $\mu(xy) = \max\{\mu(x), \mu(y)\}$, $\forall x, y \in S$.

Conversely, let $\mu(xy) = \max\{\mu(x), \mu(y)\}$, $\forall x, y \in S$. Suppose $x_r \circ y_r \in \mu$. Then $(xy)_r = x_r \circ y_r \in \mu$. So $r \leq \mu(xy)$. If possible, suppose x_r and y_r do not belong to μ. So $r > \mu(x)$ and $r > \mu(y)$. Therefore, $r > \max\{\mu(x), \mu(y)\}$ i.e. $r > \mu(xy)$, a contradiction. Hence $x_r \in \mu$ or $y_r \in \mu$. Thus μ is a weakly completely prime fuzzy ideal of S.

Theorem 5.5 Let μ be a fuzzy ideal of a semiring S. Then μ is weakly completely prime if and only if μ_t $(0 < t \leq 1)$ is a completely prime ideal of S for $\mu_t \neq \varphi$.

Proof: Suppose μ is a weakly completely prime fuzzy ideal of S and $xy \in \mu_t$. Then $\mu(xy) = \max\{\mu(x), \mu(y)\} \geq t$. Therefore, $\mu(x) \geq t$ or $\mu(y) \geq t$. Hence $x \in \mu_t$ or $y \in \mu_t$. Thus μ_t is a completely prime ideal of S.

Conversely, let μ_t be a completely prime ideal of S, where $t \in (0, 1]$. Suppose $x, y \in S$ and $\mu(xy) = t$. Then $xy \in \mu_t$. Since μt is a completely prime ideal of S, $x \in \mu_t$ or $y \in \mu_t$. Therefore, $\mu(x) \geq t$ or $\mu(y) \geq t$. So $\max\{\mu(x), \mu(y)\} \geq t$ i.e. $\mu(xy) \leq \max\{\mu(x), \mu(y)\}$. Also since μ is a fuzzy ideal of S, $\mu(xy) \geq \max\{\mu(x), \mu(y)\}$. Thus $\mu(xy) = \max\{\mu(x), \mu(y)\}$. Hence by Theorem 5.4, μ is a weakly completely prime fuzzy ideal of S.

Theorem 5.6 Let S be a semiring and μ be a fuzzy ideal of S. Then the following statements hold:
(i) If μ is a completely prime fuzzy ideal of S, then μ is a prime fuzzy ideal and a weakly completely prime fuzzy ideal of S.
(ii) If μ is a prime fuzzy ideal of S, then μ is a weakly prime fuzzy ideal of S.
(iii) If μ is a weakly completely prime fuzzy ideal of S, then μ is a weakly prime fuzzy ideal of S.

Proof: (i) Suppose μ is a completely prime fuzzy ideal of S, then by the definition of weakly completely prime fuzzy ideal, μ is a weakly completely prime fuzzy ideal of S and by Theorem 5.3, μ is a prime fuzzy ideal of S.
(ii) Obvious.
(iii) Suppose μ is a weakly completely prime fuzzy ideal of S and $tC_H \circ tC_K \subseteq \mu$, where H, K are ideals of S and $t \in (0, 1]$. So $tC_{HK} \subseteq \mu$. If possible, let $tC_H\, tC_K$ be not subset of μ, then there exist $x \in H$, $y \in K$ such that $\mu(x) < t$, $\mu(y) < t$. Since μ is a weakly completely prime fuzzy ideal, by Theorem 5.4, $\mu(xy) = \max\{\mu(x), \mu(y)\} < t$ i.e. $(xy)_t$ does not belong to μ, a contradiction. Thus μ is a weakly prime fuzzy ideal of S.

Theorem 5.7 Let S be a commutative semiring and μ be a fuzzy ideal of S. Then
(i) μ is a completely prime fuzzy ideal of S if and only if μ is a prime fuzzy ideal of S.
(ii) μ is a weakly completely prime fuzzy of S if and only if μ is a weakly prime fuzzy ideal of S.

Proof: (i) Suppose μ is a prime fuzzy ideal of S and $x_r \circ y_t \in \mu$, where $x, y \in S$, $r, t \in (0, 1]$.
Let $p = \min\{r, t\}$. Since S is commutative, we have $< x_r > \circ < y_t > = pC_{<x><y>} \subseteq pC_{<xy>} = < (xy)_p > = < x_r \circ y_t >$ (by Proposition 2.14(ii)). So $< x_r > \circ < y_t > \subseteq < x_r \circ y_t > \subseteq \mu$. Since μ is a prime fuzzy ideal of S, $< x_r > \subseteq \mu$ or $< y_t > \subseteq \mu$ i.e. $x_r \in \mu$ or $y_t \in \mu$. Hence μ is a completely prime fuzzy ideal of S.

The converse part is clear.

(ii) Let μ be a weakly prime fuzzy ideal of S and x_r o $y_r \in \mu$, where x, y $\in$ S, r $\in$ (0, 1]. Since S is commutative, we have $rC_{<x>}$ o $rC_{<y>} = rC_{<x><y>} \subseteq rC_{<xy>} =< (xy)_r >=< x_r$ o $y_r >$. So $rC_{<x>}$ o $rC_{<y>} \subseteq< x_r$ o $y_r >\subseteq \mu$. Since μ is a weakly prime fuzzy ideal of S, $rC_{<x>} \subseteq \mu$ or $rC_{<y>} \subseteq \mu$ i.e. $x_r \in \mu$ or $y_r \in \mu$. Hence μ is a weakly completely prime fuzzy ideal of S.

The converse part is obvious.

6. Quasi-prime Fuzzy Left Ideals

Definition 6.1 A fuzzy left ideal μ is called quasi-prime if for any two fuzzy left ideals μ_1 and μ_2 of S, μ_1 o $\mu_2 \subseteq \mu$ implies that $\mu_1 \subseteq \mu$ or $\mu_2 \subseteq \mu$.

Theorem 6.2 A fuzzy left ideal μ of S is quasi-prime if and only if for any two fuzzy points x_r, y_t of S (r, t $\in$ (0, 1]), x_r o S o $y_t \subseteq \mu$ implies that $x_r \in \mu$ or $y_t \in \mu$.

Proof: Let μ be a quasi-prime fuzzy left ideal of S and x_r o S o $y_t \subseteq \mu$, where xr and yt are two fuzzy points of S. So S o x_r o S o $y_t \subseteq$ S o $\mu \subseteq \mu$. Since (S o x_r), (S o y_t) are fuzzy left ideals of S, μ is a quasi-prime left ideal of S and (S o x_r) o (S o y_t) $\subseteq \mu$, S o $x_r \subseteq \mu$ or S o $y_t \subseteq \mu$. Therefore $(< x_r >_L)^2 \subseteq \mu$ or $(< y_t >_L)^2 \subseteq \mu$. So $< x_r >_L \subseteq \mu$ or $< y_t >_L \subseteq \mu$ i.e. $x_r \in \mu$ or $y_t \in \mu$.

Conversely, suppose the condition holds and μ_1 o $\mu_2 \subseteq \mu$, where μ_1, μ_2 are two fuzzy left ideals of S. If possible, let μ_1 and μ_2 be not subsets of μ, so there exist x, y $\in$ S such that $\mu_1(x) > \mu(x)$ and $\mu_2(y) > \mu(y)$. Let $\mu_1(x) = r$ and $\mu_2(y) = t$. So $x_r \in \mu_1$ and $y_t \in \mu_2$ but x_r, y_t do not belong to μ. Now x_r o S o $y_t \subseteq \mu_1$ o $\mu_2 \subseteq \mu$. Therefore, by given condition $x_r \in \mu$ or $y_t \in \mu$, a contradiction. So either $\mu_1 \subseteq \mu$ or $\mu_2 \subseteq \mu$. Hence μ is a quasi-prime fuzzy left ideal of S.

Proposition 6.3 A left ideal L of a semiring S is a quasi-prime if and only if C_L is a quasi-prime fuzzy left ideal of S.

Proof: Let L be a quasi-prime left ideal of S. Suppose μ_1 and μ_2 are two fuzzy left ideals of S such that $\mu_1 o \mu_2 \subseteq C_L$. If possible, let μ_1 and μ_2 be not subset of C_L. Then there exist x, y $\in$ S such that $\mu_1(x) > C_L(x)$ and $\mu_2(y) > C_L(y)$. So $\mu_1(x) > 0$, $\mu_2(y) > 0$ and $C_L(x) = C_L(y) = 0$. This implies x, y do not belong to L. Then SxSy is not subset of L, otherwise since L is a quasi-prime left ideal of S, (Sx)(Sy) $\subseteq$ L $\Rightarrow$ Sx $\subseteq$ L or Sy $\subseteq$ L $\Rightarrow < x >_L \subseteq$ L or $< y >_L \subseteq$ L $\Rightarrow$ x $\in$ L or y $\in$ L, a contradiction. Therefore, there exist r_1, $r_2 \in$ S such that r_1xr_2y does not belong to L. Let $a = r_1xr_2y$. Then $C_L(a) = 0$ and $\mu_1 o \mu_2(a) = \sup_{a=cd} \min\{\mu_1(c), \mu_2(d)\} \geq$ $\min\{\mu_1(r_1x), \mu_2(r_2y)\} \geq \{\mu_1(x), \mu_2(y)\} > 0$ [since μ_1 and μ_2 are fuzzy left ideals], which contradicts the fact $\mu_1 o \mu_2 \subseteq C_L$. Hence $\mu_1 o \mu_2 \subseteq C_L \Rightarrow \mu_1 \subseteq C_L$ or $\mu_2 \subseteq C_L$. Thus C_L is a quasi-prime fuzzy left ideal of S.

Again, let C_L be a quasi-prime fuzzy left ideal of S, where L is a left ideal of S. Suppose H, K are two left ideals of S and HK $\subseteq$ L. So C_H o $C_K = C_{HK} \subseteq C_L$. Since C_L is a quasi-prime fuzzy left ideal of S, $C_H \subseteq C_L$ or $C_K \subseteq C_L$ which implies that either H $\subseteq$ L or K $\subseteq$ L. Hence L is a quasi-prime left ideal of S.

Definition 6.4 A fuzzy subset μ of S is called a fuzzy m-system if r, t $\in$ [0, 1) and x, y $\in$ S with $\mu(x) > r$, $\mu(y) > t$ then there exists a $\in$ S such that $\mu(xay) > \max\{r, t\}$.

Proposition 6.5 Let M be a subset of S. Then M is a m-system of S if and only if C_M is a fuzzy m-system.

Proof: Let M be a m-system of S and r, t $\in$ [0, 1), x, y $\in$ S with $C_M(x) > r$, $C_M(y) > t$, then x, y $\in$ M and $\max\{r, t\} < 1$. Since M is a m-system, there exists a $\in$ S such that xay $\in$ M. Therefore $C_M(xay) = 1 > \max\{r, t\}$. Thus C_M is a fuzzy m-system.

Again let M be a subset of S such that C_M is a fuzzy m-system. Suppose x, y ∈ M. So $C_M(x) = C_M(y) = 1$. Therefore, for any r, t ∈ [0, 1), $C_M(x) > r$, $C_M(y) > t$. Since C_M is a fuzzy m-system, there exists a ∈ S such that $C_M(xay) > \max\{r, t\}$. Since r, t ∈ [0, 1) is arbitrary, $C_M(xay) = 1$ i.e. xay ∈ M. Hence M is a m-system.

Theorem 6.6 Let μ be a fuzzy left ideal of S. Then μ is quasi-prime if and only if 1−μ is a fuzzy m-system.

Proof: Let μ be a fuzzy quasi-prime left ideal of S and r, t ∈ [0, 1). Suppose x, y ∈ S such that $(1-\mu)(x) > r$, $(1-\mu)(y) > t$. So $\mu(x) < 1-r$, $\mu(y) < 1-t$ i.e. x_{1-r}, y_{1-t} do not belong to μ. Therefore, by Theorem 6.2, there exists a ∈ S such that x_{1-r} o $C_{\{a\}}$ o $y_{1-t} = (xay)_{\min\{1-r,1-t\}}$ does not belong to μ. So $\mu(xay) < \min\{1 - r, 1 - t\} = 1 - \max\{r, t\}$. Thus $(1 - \mu)(xay) > \max\{r, t\}$. Hence 1 – μ is a fuzzy m-system.
Again let 1−μ be a fuzzy m-system of S and x_r o S o $y_t \subseteq \mu$, where x_r and y_t are two fuzzy points of S (r, t ∈ (0, 1]). If possible, suppose x_r and y_t do not belong to μ. So $\mu(x) < r$ and $\mu(y) < t$ i.e. $(1-\mu)(x) > 1-r$ and $(1-\mu)(y) > 1-t$. Since 1−μ is a fuzzy m-system, there exists a ∈ S such that $(1-\mu)(xay) > \max\{1-r, 1-t\} = 1-\min\{r, t\}$. Thus $\mu(xay) < \min\{r, t\}$, which implies that x_{1-r} o $C_{\{a\}}$ o $y_{1-t} = (xay)_{\min\{1-r,1-t\}}$ does not belong to μ, a contradiction. Thus x_r o S o $y_t \subseteq \mu$ implies that $x_r \in \mu$ or $y_t \in \mu$. Therefore, by Theorem 6.2, μ is a quasi-prime fuzzy left ideal of S.

7. Weakly Quasi-prime Fuzzy Left Ideals

Definition 7.1 A fuzzy left ideal μ of a semiring S is called weakly quasi-prime if μ_1 and μ_2 are two fuzzy left ideals of S with $\mu \subseteq \mu_1$, $\mu \subseteq \mu_2$ and μ_1 o $\mu_2 \subseteq \mu$ then $\mu_1 \subseteq \mu$ or $\mu_2 \subseteq \mu$.

Proposition 7.2 A left ideal L of a semiring S is a weakly quasi-prime if and only if C_L is a weakly quasi-prime fuzzy left ideal of S.

Proof: Let L be a weakly quasi-prime left ideal of S. Clearly, C_L is a fuzzy left ideal of S. Suppose μ_1 and μ_2 are two fuzzy left ideals of S with $C_L \subseteq \mu_1$, $C_L \subseteq \mu_2$ and μ_1 o $\mu_2 \subseteq C_L$. If possible, let μ_1 and μ_2 be not subsets of C_L. Then there exist x, y ∈ S such that $\mu_1(x) > C_L(x)$ and $\mu_2(y) > C_L(y)$. So x, y do not belong to L and $\mu_1(x) > 0$, $\mu_2(y) > 0$. We now prove that (Sx ∪ L) ∪ (Sy ∪ L) is not subset of L. If possible, let (Sx ∪ L) ∪ (Sy ∪ L) ⊆ L. As L is a weakly quasi-prime left ideal of S and Sx∪L, Sy ∪L are left ideals of S containing L, either Sx ∪ L = L or Sy ∪ L = L. So Sx ⊆ L or Sy ⊆ L. Therefore $< x >_L \subseteq L$ or $< y >_L \subseteq L$, which implies that either x ∈ L or y ∈ L, a contradiction. Thus (Sx ∪ L) ∪ (Sy ∪ L) is not a subset of L.
Then there exist $s_1, s_2 \in S$ such that $(\{s_1x\} \cup L)(\{s_2y\} \cup L)$ is not a subset of L. Therefore, s_1xs_2y does not belong to L or Ls_2y is not a subset of L.
If s_1xs_2y does not belong to L then $C_L(s_1xs_2y) = 0$ and μ_1 o $\mu_2(s_1xs_2y) \geq \min\{\mu_1(s_1x), \mu_2(s_2y)\} \geq \min\{\mu_1(x), \mu_2(y)\} > 0$, a contradiction, since μ_1 o $\mu_2 \subseteq C_L$.
If Ls_2y is not a subset of L, then there exists l ∈ L such that ls_2y does not belong to L. Therefore $C_L(ls_2y) = 0$ but μ_1 o $\mu_2(ls_2y) \geq \min\{\mu_1(l), \mu_2(s_2y)\} \geq \min\{C_L(l), \mu_2(y)\}$[since $C_L \subseteq \mu_1$] $= \mu_2(y) > 0$, a contradiction.
Hence $\mu_1 \subseteq C_L$ or $\mu_2 \subseteq C_L$ i.e. C_L is a weakly quasi-prime fuzzy left ideal of S.

Conversely, let C_L be a weakly quasi-prime fuzzy left ideal of S and H, K be two left ideals of S with L ⊆ H, L ⊆ K and HK ⊆ L. Then $C_L \subseteq CH$, $C_L \subseteq CK$ and C_H o $C_K = C_{HK} \subseteq C_L$. Since C_L is a weakly quasi-prime fuzzy left ideal of S, either $C_H \subseteq C_L$ or $C_K \subseteq C_L$ i.e. H ⊆ L or K ⊆ L. Hence L is a weakly quasi-prime left ideal of S.

Theorem 7.3 Let S be a semiring and μ be a fuzzy left ideal of S. Then the following statements are equivalent.
(i) μ is weakly quasi-prime fuzzy left ideal of S.
(ii) $(\mu_1 \cup \mu)$ o $(\mu_2 \cup \mu) \subseteq \mu$ implies $\mu_1 \subseteq \mu$ or $\mu_2 \subseteq \mu$, where μ_1, μ_2 are two fuzzy left ideals of S.
(iii) $\mu \subseteq \mu_1$ and μ_1 o $\mu_2 \subseteq \mu$ imply $\mu_1 = \mu$ or $\mu_2 \subseteq \mu$, where μ_1, μ_2 are two fuzzy left ideals of S.

(iv) $(\mu_1 \cup \mu) o \mu_2 \subseteq \mu$ implies $\mu_1 \subseteq \mu$ or $\mu_2 \subseteq \mu$, where μ_1, μ_2 are two fuzzy left ideals of S.
(v) For any two fuzzy points x_r, y_t of S (r, t ∈ (0, 1]), $(x_r \cup \mu) o S o (y_t \cup \mu) \subseteq \mu$ implies $x_r \in \mu$ or $y_t \in \mu$.

Proof: (i) ⇒ (ii) Let μ_1, μ_2 be two fuzzy left ideals of S such that $(\mu_1 \cup \mu) o (\mu_2 \cup \mu) \subseteq \mu$.
Then $\mu_1 \cup \mu$, $\mu_2 \cup \mu$ are two fuzzy left ideals of S with $\mu \subseteq \mu_1 \cup \mu$, $\mu \subseteq \mu_2 \cup \mu$ and $(\mu_1 \cup \mu) o (\mu_2 \cup \mu) \subseteq \mu$. Since μ is a weakly quasi-prime fuzzy left ideal of S, $\mu_1 \cup \mu \subseteq \mu$ or $\mu_2 \cup \mu \subseteq \mu$ i.e. $\mu_1 \subseteq \mu$ or $\mu_2 \subseteq \mu$.

(ii) ⇒ (iii) Suppose $\mu \subseteq \mu_1$ and $\mu_1 o \mu_2 \subseteq \mu$, where μ_1, μ_2 are two fuzzy left ideals of S.
$(\mu_1 \cup \mu) o (\mu_2 \cup \mu) \subseteq \mu_1 o (\mu_2 \cup \mu) = \mu_1 o \mu_2 \cup \mu_1 o \mu$ (by Proposition 2.12) $\subseteq \mu \cup \mu_1 o \mu \subseteq \mu$
[as μ is a fuzzy left ideal of S, $\mu_1 o \mu \subseteq \mu$]. By (ii), $\mu_1 \subseteq \mu$ or $\mu_2 \subseteq \mu$. Since $\mu \subseteq \mu_1$, $\mu_1 = \mu$ or $\mu_2 \subseteq \mu$.

(iii) ⇒ (iv) Suppose $(\mu_1 \cup \mu) o \mu_2 \subseteq \mu$, where μ_1, μ_2 are two fuzzy left ideals of S. So by (iii), $\mu_1 \cup \mu = \mu$ or $\mu_2 \subseteq \mu$ i.e. $\mu_1 \subseteq \mu$ or $\mu_2 \subseteq \mu$.

(iv) ⇒ (v) Let $(x_r \cup \mu) o S o (y_t \cup \mu) \subseteq \mu$, where x_r, y_t are two fuzzy points of S (r, t ∈ (0, 1]). Now $x_r o S o y_t \subseteq (x_r \cup \mu) o S o (y_t \cup \mu) \subseteq \mu$ and $\mu o S o y_t \subseteq (x_r \cup \mu) o S o (y_t \cup \mu) \subseteq \mu$. So $(x_r \cup S o x_r \cup \mu) o (S o y_t) = x_r o S o y_t \cup S o xr o S o y_t \cup \mu o S o y_t$ (by Proposition 2.12) $\subseteq \mu \cup S o \mu \cup \mu \subseteq \mu$. As $x_r \cup S o x_r = < x_r >_L$ and $S o y_t$ are two fuzzy left ideals of S, by (iv), $< x_r >_L \subseteq \mu$ or $S o y_t \subseteq \mu$.
If $< x_r >_L \subseteq \mu$, then $x_r \in < x_r >_L \subseteq \mu$.
If $S o y_t \subseteq \mu$, then $(< y_t >_L \cup \mu) o < y_t >_L = (< y_t >_L)^2 \cup \mu o < y_t >_L$(by Proposition 2.12) $\subseteq S o y_t \subseteq \mu$.
So by (iv), $< y_t >_L \subseteq \mu$ i.e. $y_t \in < y_t >_L \subseteq \mu$.

(v) ⇒ (i) Let μ_1 and μ_2 be two fuzzy left ideals of S such that $\mu \subseteq \mu_1$, $\mu \subseteq \mu_2$ and $\mu_1 o \mu_2 \subseteq \mu$. If possible, let μ_1 and μ_2 be not subsets of μ, then there exist x, y ∈ S such that $\mu_1(x) > \mu(x)$ and $\mu_2(y) > \mu(y)$. Suppose $\mu(x) = r$ and $\mu_2(y) = t$. Then x_r, y_t do not belong to μ but $x_r \in \mu_1$, $y_t \in \mu_2$ (r, t ∈ (0, 1]).
Now $(x_r \cup \mu) o S o (y_t \cup \mu) \subseteq \mu_1 o S o \mu_2 \subseteq \mu_1 o \mu_2 \subseteq \mu$. Therefore, by (v), $x_r \in \mu$ or $y_t \in \mu$, a contradiction. Hence $\mu_1 \subseteq \mu$ or $\mu_2 \subseteq \mu$. Thus μ is a weakly quasi-prime fuzzy left ideal of S.

Theorem 7.4 Let S be a commutative semiring. Then the following statements are equivalent:
(i) μ is a prime fuzzy left ideal of S.
(ii) μ is a quasi-prime fuzzy left ideal of S.
(iii) μ is a weakly quasi-prime fuzzy left ideal of S.

Proof: (i) ⇒ (ii) and (ii) ⇒ (iii) are obvious.
(iii) ⇒ (i) Let μ be a weakly quasi-prime fuzzy left ideal of S. Suppose $\mu_1 o \mu_2 \subseteq \mu$, where μ_1, μ_2 are fuzzy ideals of S. Since S is commutative, μ is a fuzzy ideal of S. Now $\mu \subseteq \mu_1 \cup \mu$, $\mu \subseteq \mu_2 \cup \mu$ and $(\mu_1 \cup \mu) o (\mu_2 \cup \mu) \subseteq \mu_1 o \mu_2 \cup \mu_1 o \mu \cup \mu o \mu_2 \cup \mu_2 \subseteq \mu$. As μ is a weakly quasi-prime left ideal of S, $\mu_1 o \mu \subseteq \mu$ or $\mu_2 o \mu \subseteq \mu$ i.e. $\mu_1 \subseteq \mu$ or $\mu_2 \subseteq \mu$. Hence μ is a fuzzy prime ideal of S.

Theorem 7.5 Let S be a semiring, μ be a fuzzy left ideal and ν be a fuzzy subset of S satisfying the following conditions:
(i) $\mu \cap \nu = \varphi$.
(ii) For any two fuzzy points $x_r, y_t \in \nu$, $(x_r \cup \mu) o S o (y_t \cup \mu) \cap \nu \neq \varphi$.
If λ is a maximal fuzzy left ideal of S with respect to containing μ and $\lambda \cap \nu = \varphi$, then λ is a weakly quasi-prime fuzzy left ideal of S.

Proof: Let μ_1, μ_2 be two fuzzy left ideals of S such that $\lambda \subseteq \mu_1, \mu_2$ and $\mu_1 o \mu_2 \subseteq \lambda$. If possible, let λ be a proper subset of μ_1 and μ_2. Then there exist fuzzy points $x_r \in \mu_1$, $y_t \in \mu_2$ but x_r, y_t do not belong to λ. By maximality of λ, we get $(\lambda \cup < x_r >_L) \cap \nu \neq \varphi$ and $(\lambda \cup < y_t >_L) \cap \nu \neq \varphi$.
Let $a_l \in (\lambda \cup < x_r >_L) \cap \nu \neq \varphi$ and $b_m \in (\lambda \cup < y_t >_L) \cap \nu \neq \varphi$, (l,m ∈ (0, 1]). Now $(a_l \cup \mu) o S o (b_m \cup \mu) \subseteq (\mu_1 \cup \mu) o S o (\mu_2 \cup \mu)$
$\subseteq \mu_1 o S o \mu_2 \cup \mu_1 o S o \mu \cup \mu o S o \mu_2 \cup \mu o S o \mu$

$\subseteq \mu_1 \text{ o } \mu_2 \cup \mu 1 \text{ o } \mu \cup \mu \text{ o } \mu_2 \cup \mu$
$\subseteq \lambda \cup \mu_1 \text{ o } \lambda \cup \lambda \text{ o } \mu_2 \cup \lambda$, since $\mu \subseteq \lambda$
$\subseteq \lambda$.
Since $\lambda \cap \nu = \varphi$, $(a_l \cup \mu) \text{ o } S \text{ o } (b_m \cup \mu) \cap \nu = \varphi$, a contradiction. Therefore, either $\mu_1 \subseteq \lambda$ or $\mu_2 \subseteq \lambda$. Hence λ is a weakly quasi-prime fuzzy left ideal of S.

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