L-fuzzy soft groups

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Abstract

The present paper aims to introduce L-fuzzy soft groups and studies some of its properties. For this purpose, we define L-fuzzy soft sets, image and pre image of L-fuzzy soft sets under soft mapping [8] and study the behaviour of soft homomorphic image and soft homomorphic pre image of L-fuzzy soft groups. Fundamental homomorphism theorem is established in L-fuzzy soft set setting.

Key words: Soft sets, soft groups, fuzzy soft sets, fuzzy soft groups, intuitionist fuzzy set, L-fuzzy set, L-fuzzy subgroups, L-fuzzy Soft Sets, L-fuzzy soft groups

1. Introduction

There are several techniques to represent and solve various types of uncertainties prevailing in this physical world. There are theories viz. theories of probability, theory of fuzzy sets, theory of multi sets, theory of rough sets, theory of vague sets etc. which can handle uncertainties of various types. Molodtsov (1999) initiated the theory of soft sets as a new mathematical tool for dealing with uncertainties which traditional mathematical tools cannot handle. He has shown several applications of this theory in solving many practical problems in economics, engineering, social science, medical science etc. Later other authors like Maji, Biswas and Roy (2003) have further studied the theory of soft sets and used this theory to solve some decision making problems. They have also introduced the concept of fuzzy soft sets (Maji, Biswas & Roy, 2001), a more generalized concept, which is a combination of fuzzy set and soft set and studied its properties. Ever since mathematical activities are going on with all these concepts individually as well as their hybridizations such as fuzzy subgroups (Rosenfeld, 1971), soft group (Aktas & Cagman, 2007; Aslam & Qurashi, 2012; Nazmul & Samanta, 2011), fuzzy soft groups (Nazmul & Samanta, 2011), soft mappings (Kharal & Ahmad; Majumdar & Samanta, 2010) soft topology (Cagman, Karatas & Enginoglu, 2011; Hazra, Majumdar & Samanta, 2012; Shabir & Naz, 2011) soft topological group (Nazmul & Samanta, 2012; Nazmul & Samanta, 2015; Nazmul & Samanta, 2014) etc. Also it is noticed that, in the discussion of functional image and pre image of soft groups and fuzzy soft groups, the corresponding authors have considered actually the crisp mappings. They have considered a special type of soft mappings (Kharal & Ahmad), where the parameter set in the image and pre image soft sets remain unchanged as in the main soft sets i.e. they have considered the soft mapping \( f_\varphi \),

where \( \varphi: A \to A \), defined by \( \varphi(\alpha) = \alpha, \forall \alpha \in A \). As a natural continuation of this we have introduced in this paper, a notions of L-fuzzy soft groups. Its subsystems viz. L-fuzzy soft subgroups, L-fuzzy soft normal subgroups are defined and study the properties of soft homomorphic image and pre image of L-fuzzy soft groups under soft mappings \( f_\varphi \), where both the mappings \( f: X \to Y, \varphi: A \to B \) are arbitrary. Finally L-fuzzy soft group version of the celebrated isomorphism theorem is established.
2. Preliminaries

Following (Atanassov & Stoeva, 1984; Feng; Goguen, 1967; Maji, Biswas & Roy, 2003; Molodtsov, 1999), some definitions and results regarding L-fuzzy sets, L-fuzzy subgroups, soft sets and fuzzy soft sets are given in this section which will be used in the main result of this paper.

Definition 2.1. (Atanassov & Stoeva, 1984; Feng; Goguen, 1967) Let $X$ be a non-empty set and $L$ will denote completely distributive lattice with order-reversing involution. A L-fuzzy set in $X$ is an element of the set $L^X$ of all functions from $X$ into the lattice $L$. Let $L(X)$ denote the set of all L-fuzzy subsets of $X$.

Definition 2.2. [6] Let $f : (X, L_X) \rightarrow (Y, L_Y)$ be a mapping and $\mu \in L(X), \nu \in L(Y)$. Define the image of $f(\mu)$ and the pre image $f^{-1}(\nu)$ by

\[(f(\mu))(y) = \begin{cases} \mu(x); & f(x) = y, \ x \in X \ \text{if} \ f^{-1}(y) \neq \phi, \ \forall \ y \in Y; \\ 0 & \text{if otherwise} \end{cases} \]

and $[f^{-1}(\nu)](x) = \nu[f(x)], \ \forall \ x \in X$.

Definition 2.3. (Feng; Palaniappan, Naganathan & Arjunan, 2009) An L-fuzzy group (subgroup) $\mathcal{L}$ on a group $X$ is an L-fuzzy set in $X$ satisfying the following conditions:

\[(i) \ \mathcal{L}(x_1x_2) \geq \mathcal{L}(x_1) \wedge \mathcal{L}(x_2), \ \forall \ x_1, x_2 \in X, \ \text{and} \ (ii) \ \mathcal{L}(x^{-1}) \geq \mathcal{L}(x), \ \forall \ x \in X.\]

It is clear that if $\mathcal{L}$ is an L-fuzzy subgroup (LG for short) of a group $X$ with identity element $e_X$, then $\mathcal{L}(x^{-1}) = \mathcal{L}(x)$, and $\mathcal{L}(x) \leq \mathcal{L}(e_X)$, $\forall \ x \in X$.

Proposition 2.4. (Feng) $\mathcal{L}$ is an L-fuzzy subgroup of a group $X$ iff $\mathcal{L}(x_1x_2^{-1}) \geq \mathcal{L}(x_1) \wedge \mathcal{L}(x_2), \ \forall \ x_1, x_2 \in X$.

Proposition 2.5. (Feng) Let $\mathcal{L}$ be an L-fuzzy set on a group $X$, then $\mathcal{L}$ is an LG of $X$ iff $\mathcal{L}_l = \{x \in X: \mathcal{L}(x) \geq l\}$ are subgroups of $X, \forall l \in L$.

Proposition 2.6. (Feng) Let $X$ and $Y$ be two groups and $f : X \rightarrow Y$ be a homomorphism. If $\mathcal{L}$ (or $\mathcal{M}$) is an LG on $X$(or $Y$) then $f(\mathcal{L})$(or $f(\mathcal{M})$) is an LG on $Y$(or $X$).

Definition 2.7. (Nazmul & Samanta, 2011) Let $X$ be a group and $A$ be a set of parameters. $FP(X)$ denotes the set of all fuzzy subsets of $X$. Let $(F, A)$ be a fuzzy soft set over $X$. Then $(F, A)$ is said to be a fuzzy soft group over $X$, "a $\tilde{A}$ $\tilde{A}$".

Definition 2.8. (Maji, Biswas & Roy, 2011; Molodtsov, 1999) A pair $(F, A)$ is called

(i) a soft set over $X$, where $F$ is a mapping given by $F : A \rightarrow P(X)$.

(ii) a fuzzy soft set over $X$, where $F$ is a mapping given by $F : A \rightarrow FP(X)$.

Let $\{(F_i, A_i) : i \in D\}$ be a nonempty family of soft sets (fuzzy soft sets) over a common universe $X$. Then their

(i) $\Lambda ND$ denoted by $\Lambda_{i \in D}$, is defined by $\Lambda_{i \in D}(F_i, A) = (\Lambda_{i \in D}F_i, A^\Delta)$, where $\Lambda_{i \in D}F_i(\alpha) = \cap_{i \in D}(F_i(\alpha_i)), \forall \alpha = (\alpha_i) \in A^\Delta$ is a soft sets (fuzzy soft sets) over $X$. 

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**Definition 2.9.** [8] Let $S(X, A)$ and $S(Y, B)$ be the families of all soft sets over $X$ and $Y$ respectively. The mapping $f : S(X, A) \rightarrow S(Y, B)$ is called a soft mapping from $X$ to $Y$, where $f : X \rightarrow Y$ and $j : A \rightarrow \mathcal{B}$ are two mappings. Also

(i) the image of a soft set $(F, A) \in S(X, A)$ under the mapping $f_j$ is denoted by $f_j[(F, A)]$, and is defined by $f_j[(F, A)] = \{(f(F(\alpha)), j(\alpha)) \mid \alpha \in A\}$.

(ii) the inverse image of a soft set $(G, B) \in S(Y, B)$ under the mapping $f_j$ is denoted by $f_j^{-1}[(G, B)] = (f^{-1}(G), A)$, and is defined by $f_j^{-1}[(G, B)](\alpha) = f^{-1}(G(j(\alpha)))$, \(\forall \alpha \in A\).

(iii) The soft mapping $f_j$ is called injective (surjective) if $f$ and $j$ are both injective (surjective).

(iv) The soft mapping $f_j$ is identity soft mapping, if $f$ and $j$ are both classical identity mappings.

3. L-fuzzy soft sets

In this section we define L-fuzzy soft sets and study some of their properties where $L$ is a completely distributive lattice with an involutive order reversing operation and whose maximal and minimal elements are 1 and 0 respectively. Unless otherwise stated, $X$ will be assumed to be an initial universal set and $A$ will be taken to be a set of parameters. Let $L(X)$ denote the set of all L-fuzzy sets of $X$.

**Definition 3.1.** A pair $(F, A)$ is called an L-fuzzy soft set over $X$, where $F : A \rightarrow L(X)$.

**Example 3.2.** Let $X = \{h_1, h_2, h_3, h_4\}$, $A = \{\alpha_1, \alpha_2, \alpha_3\}$, $L = \{0, a, b, 1\}$, a diamond shaped lattice and a mapping $F : A \rightarrow L(X)$ is defined by $F(\alpha_1) = \{h_1, 1\}$, $F(\alpha_2) = \{h_2, 0\}$, $F(\alpha_3) = \{h_3, 1\}$. Then $(F, A)$ is an L-fuzzy soft set over $X$.

**Definition 3.3.** Let $(F, A)$, $(F_1, A)$ and $(F_2, A)$ be L-fuzzy soft sets over a common universe $X$. Then

(i) $(F_1, A)$ is said to be an L-fuzzy soft subset of $(F_2, A)$ if $F_1(\alpha) \subseteq F_2(\alpha)$, \(\forall \alpha \in A\). This relation is denoted by $(F_1, A) \subseteq (F_2, A)$.

(ii) $(F_1, A)$ is said to be L-fuzzy soft equal to $(F_2, A)$ if $F_1(\alpha) = F_2(\alpha)$, \(\forall \alpha \in A\). This relation is denoted by $(F_1, A) = (F_2, A)$.

(iii) the complement of $(F, A)$ is defined as $(F, A)^c = (F^c, A)$, where $F^c(\alpha) = [F(\alpha)]^c$, \(\forall \alpha \in A\).

(iv) $(F, A)$ is said to be a null L-fuzzy soft set over $X$ if $F(\alpha) = \{x \mid x \in X\}$, \(\forall \alpha \in A\). This is denoted by $\tilde{\Phi}$.

(v) $(F, A)$ is said to be an absolute L-fuzzy soft set over $X$ if $F(\alpha) = \{x \mid x \in X\}$, \(\forall \alpha \in A\). This is denoted by $\tilde{\Phi}$.
Example 3.4. Let $X$, $A$, $L$ and $(F, A)$ be the same as in Example 3.2. Let an another mapping $G : A \rightarrow L(X)$ be defined by $G(a_1) = \{ (h_1, a), (h_2, 0), (h_3, 0), (h_4, a) \}$, $G(a_2) = \{ (h_1, 0), (h_2, 0), (h_3, 0), (h_4, b) \}$, $G(a_3) = \{ (h_1, b), (h_2, 0), (h_3, a), (h_4, 0) \}$. Then $(G, A)$ is an L-fuzzy soft set over $X$, and $(G, A)^{c} \subseteq (F, A)$.

Also the complement of $(F, A)$ is $(F^{c}, A)$ is defined by $F^{c}(a_1) = \{ (h_1, 0), (h_2, a), (h_3, b), (h_4, 0) \}$, $F^{c}(a_2) = \{ (h_1, 1), (h_2, b), (h_3, a), (h_4, 0) \}$, $F^{c}(a_3) = \{ (h_1, a), (h_2, a), (h_3, a), (h_4, b) \}$.

Definition 3.5. Let $\{ (F_i, A) : i \in D \}$ be a nonempty family of L-fuzzy soft sets over a common universe $X$, then their

(i) AND, denoted by $\tilde{\wedge}_{i \in D} (F_i)$, is defined by $\tilde{\wedge}_{i \in D} (F_i, A) = (\tilde{\wedge}_{i \in D} F_i, A^\Delta)$, where $(\tilde{\wedge}_{i \in D} F_i)(a) = \cap_{i \in D} (F_i(a))$, $\forall \ a = (a_i) \in A^\Delta$ is an L-fuzzy soft set over $X$.

(ii) OR, denoted by $\tilde{\vee}_{i \in D} (F_i)$, is defined by $\tilde{\vee}_{i \in D} (F_i, A) = (\tilde{\vee}_{i \in D} F_i, A^\Delta)$, where $(\tilde{\vee}_{i \in D} F_i)(a) = \cup_{i \in D} (F_i(a))$, $\forall \ a = (a_i) \in A^\Delta$ is an L-fuzzy soft set over $X$.

(iii) Intersection, denoted by $\tilde{\cap}_{i \in D} (F_i)$, is defined by $\tilde{\cap}_{i \in D} (F_i, A) = (\tilde{\cap}_{i \in D} F_i, A)$, where $(\tilde{\cap}_{i \in D} F_i)(a) = \cap_{i \in D} (F_i(a))$, $\forall \ a \in A$ is an L-fuzzy soft set over $X$.

(iv) Union, denoted by $\tilde{\cup}_{i \in D} (F_i)$, is defined by $\tilde{\cup}_{i \in D} (F_i, A) = (\tilde{\cup}_{i \in D} F_i, A)$, where $(\tilde{\cup}_{i \in D} F_i)(a) = \cup_{i \in D} (F_i(a))$, $\forall \ a \in A$ is an L-fuzzy soft set over $X$.

Example 3.6. Let $X$, $A$ and $(F, A)$ be same as in the Example 3.2 and $(G, A)$ be as in Example 3.4.

Therefore

(i) $(F \tilde{\wedge} G, A \times A)$ is an L-fuzzy soft set where

$(F \tilde{\wedge} G)(a_1, a_1) = \{ (h_1, 1), (h_2, b), (h_3, 0), (h_4, a) \}$,
$(F \tilde{\wedge} G)(a_1, a_2) = \{ (h_1, 0), (h_2, 0), (h_3, 0), (h_4, 0) \}$,
$(F \tilde{\wedge} G)(a_2, a_1) = \{ (h_1, 0), (h_2, 0), (h_3, 0), (h_4, a) \}$,
$(F \tilde{\wedge} G)(a_2, a_2) = \{ (h_1, 0), (h_2, 0), (h_3, 0), (h_4, b) \}$.

(ii) $(F \tilde{\vee} G, A)$ is an L-fuzzy soft set where

$(F \tilde{\vee} G)(a_1, a_1) = \{ (h_1, 1), (h_2, b), (h_3, 0), (h_4, a) \}$,
$(F \tilde{\vee} G)(a_1, a_2) = \{ (h_1, 1), (h_2, b), (h_3, 0), (h_4, 1) \}$,
$(F \tilde{\vee} G)(a_2, a_1) = \{ (h_1, a), (h_2, a), (h_3, 0), (h_4, 1) \}$,
$(F \tilde{\vee} G)(a_2, a_2) = \{ (h_1, 0), (h_2, a), (h_3, 0), (h_4, 1) \}$.

(iii) $(F \tilde{\cap} G, A)$ is an L-fuzzy soft set where

$(F \tilde{\cap} G)(a_1) = \{ (h_1, a), (h_2, 0), (h_3, 0), (h_4, 1) \}$,
$(F \tilde{\cap} G)(a_2) = \{ (h_1, 0), (h_2, 0), (h_3, 0), (h_4, 1) \}$.

(iv) $(F \tilde{\cup} G, A)$ is an L-fuzzy soft set where

$(F \tilde{\cup} G)(a_1) = \{ (h_1, 1), (h_2, b), (h_3, 0), (h_4, a) \}$,
$(F \tilde{\cup} G)(a_2) = \{ (h_1, 1), (h_2, b), (h_3, 0), (h_4, 1) \}$.

Proposition 3.7. Let $(F_1, A), (F_2, A)$ be two L-fuzzy soft sets over $X$. Then

(i) $((F_1, A) \tilde{\vee} (F_2, A))^c = (F_1, A)^c \tilde{\wedge} (F_2, A)^c = (F_1^c, A) \tilde{\wedge} (F_2^c, A)$.

(ii) $((F_1, A) \tilde{\wedge} (F_2, A))^c = (F_1, A)^c \tilde{\vee} (F_2, A)^c = (F_1^c, A) \tilde{\vee} (F_2^c, A)$.

(iii) $((F_1, A) \tilde{\cap} (F_2, A))^c = (F_1, A)^c \tilde{\cap} (F_2, A)^c = (F_1^c, A) \tilde{\cap} (F_2^c, A)$.

(iv) $((F_1, A) \tilde{\cup} (F_2, A))^c = (F_1, A)^c \tilde{\cup} (F_2, A)^c = (F_1^c, A) \tilde{\cup} (F_2^c, A)$.

Proof. Proofs are straightforward.
Definition 3.8. Let \( f_j \) is a soft mapping, where \( f : X \otimes Y \) and \( j : A \otimes B \) are crisp mappings. Then

(i) the image of a L-fuzzy soft set \((F, A) \in \text{LFS}(X, A)\) under the mapping \( f_j \) is denoted by \( f_j ([F, A]) = (f_j (F), B) \), and is defined by

\[
[f_\varphi(F)](\beta) = \begin{cases} 
\bigcup_{\alpha \in \varphi^{-1}(\beta)} [f[F(\alpha)]] & \text{if } \varphi^{-1}(\beta) \neq \phi \\
\emptyset & \text{otherwise}
\end{cases}, \quad \forall \beta \in B.
\]

(ii) the inverse image of a L-fuzzy soft set \((G, B) \in \text{LFS}(Y, B)\) under the mapping \( f_j \) is denoted by \( f_j^{-1} ([G, B]) = (f_j^{-1}(G), A) \), and is defined by \([f_j^{-1}(G)(a)] = f^{-1} [G[f j (a)]]\), " \( a \in A \).

4. L-fuzzy soft groups

In this section, definitions of L-fuzzy soft groups and its subsystems viz. L-fuzzy normal soft subgroups, L-fuzzy factor soft group are introduced and study some of their properties. Notions of homomorphism and isomorphism between L-fuzzy soft groups are also introduced and the isomorphism theorem is established in L-fuzzy soft group setting. Throughout this section, \( X, Y \) are taken to be groups, \( A \) be any nonempty set of parameters and unless otherwise stated, \( L \) be a completely distributive lattice with an involutive order reversing operator and whose maximal and minimal element are 1 and 0 respectively.

Definition 4.1. Let \( X \) be a group, \( A \) be a set of parameters and \( L(X) \) denotes the set of all L-fuzzy subsets of \( X \). A L-fuzzy soft set \((F, A) \in \text{LFS}(X, A)\) is said to be an L-fuzzy soft group over \( X \), denoted by \((F, A) \in \text{LFS}(X, A)\), iff \( F(a) \) is an L-fuzzy subgroup of \( X \) i.e. \( F(a) \in \text{LFS}(X) \), " \( a \in A \).

Proposition 4.3. Let \((F_1, A) \) and \((F_2, A) \) be two L-fuzzy soft groups over \( X \). Then

(i) \((F_1, A) \lor (F_2, A) = (F_1 \lor F_2, A) \) is an L-fuzzy soft group over \( X \);
(ii) \((F_1, A) \land (F_2, A) = (F_1 \land F_2, AXA) \) is an L-fuzzy soft group over \( X \).

Proof. Since \((F_1, A) \) and \((F_2, A) \) are L-fuzzy soft groups over \( X \), then for any \( a \in A \) and \( X_1, X_2 \subseteq X \), we have

\[
[F_1(a)](x_1 x_2^{-1}) \geq [F_2(a)](x_2) \quad \text{and} \quad [F_2(a)](x_1 x_2^{-1}) \geq [F_2(a)](x_1) \land [F_2(a)](x_2)
\]

(i) Now for any \( a \in A \) and for any \( x_1, x_2 \in X \), we have

\[
\{[F_1 \lor F_2](\alpha)\}(x_1 x_2^{-1}) = [F_1(\alpha)](x_1 x_2^{-1}) \lor [F_2(\alpha)](x_1 x_2^{-1}) \geq [F_1(\alpha)](x_1) \land [F_2(\alpha)](x_2) \land [F_2(\alpha)](x_1) \land [F_2(\alpha)](x_2)
\]

Thus, \([F_1 \lor F_2](\alpha)\) is a L-fuzzy subgroup of \( X \), " \( a \in A \).

Therefore, \((F_1, A) \lor (F_2, A) \) is an L-fuzzy soft group over \( X \).
Also for any $a_1, a_2 \in A$ and $x_1, x_2 \in X$, we have
\[
\{[F_1 \wedge F_2](\alpha_1, \alpha_2)\}(x_1x_2^{-1}) = [F_1(\alpha_1)](x_1x_2^{-1}) \wedge [F_2(\alpha_2)](x_1x_2^{-1}) \\
\geq [F_1(\alpha_1)](x_1) \wedge [F_1(\alpha_1)](x_2) \wedge [F_2(\alpha_2)](x_1) \wedge [F_2(\alpha_2)](x_2) \\
= \{[F_1 \wedge F_2](\alpha_1, \alpha_2)\}(x_1) \wedge \{[F_1 \wedge F_2](\alpha_1, \alpha_2)\}(x_2)
\]

Thus, $[F_1 \wedge F_2](\alpha_1, \alpha_2)$ is a L-fuzzy subgroup of $X$, $\forall \ (a_1, a_2) \in A \times A$.

Therefore, $(F_1, A) \wedge (F_2, A)$ is an L-fuzzy soft group over $X$.

**Remark 4.4.** The union of two L-fuzzy soft groups $(F_1, A)$ and $(F_2, A)$ is not an L-fuzzy soft group in general. If $F_1(a) \wedge F_2(a)$ or $F_2(a) \wedge F_1(a), \forall \ a \in A$, then their union $(F_1, A) \wedge (F_2, A)$ is an L-fuzzy soft group over $X$.

**Definition 4.5.** Let $(F_1, A)$ and $(F_2, A)$ be two L-fuzzy soft groups over $X$. Then $(F_1, A)$ is said to be an L-fuzzy soft subgroup of $(F_2, A)$, denoted by $(F_1, A) \geq (F_2, A)$, if $F_1(a) \wedge F_2(a), \forall \ a \in A$.

**Definition 4.6.** Let $(F, A)$, $(F_1, A)$ and $(F_2, A)$ be L-fuzzy soft groups over $X$. Then $(F, A)$ is said to be
(i) identity L-fuzzy soft group over $X$ if $F(a) = 1_{\{x\}}$, $\forall \ a \in A$;
(ii) absolute L-fuzzy soft group over $X$ if $F(a) = 1_{\{X\}}, \forall \ a \in A$.

**Proposition 4.7.** Let $(F, A)$, $(F_1, A)$ and $(F_2, A)$ be L-fuzzy soft groups over $(X, A)$ and $(G, B)$, $(G_1, B)$ and $(G_2, B)$ be L-fuzzy soft groups over $(Y, B)$. Also let $f_j$ be a soft mapping.

(i) If $(F, A)$ be identity L-fuzzy soft group over $(X, A)$ and $f_j$ is a soft homomorphism such that $\varphi$ is onto, then $f_j[(F, A)]$ is identity L-fuzzy soft group over $(Y, B)$.
(ii) If $(G, B)$ be identity L-fuzzy soft group over $(Y, B)$ and $f_j$ is a soft homomorphism such that $\ker(f) = \{e_X\}$, then $f_j^{-1}[(G, B)]$ is identity L-fuzzy soft group over $(X, A)$.
(iii) If $(F, A)$ be absolute L-fuzzy soft group over $(X, A)$ and $f_j$ is onto soft mapping, then $f_j[(F, A)]$ is absolute L-fuzzy soft group over $(Y, B)$.
(iv) If $(G, B)$ be absolute L-fuzzy soft group over $(Y, B)$ and $f_j$ is a soft mapping, then $f_j^{-1}[(G, B)]$ is absolute L-fuzzy soft group over $(X, A)$.
(v) If $(G, B)$ be a L-fuzzy soft group over $(Y, B)$ and $f_j$ is a soft homomorphism, then $f_j^{-1}[(G, B)]$ is a L-fuzzy soft group over $(X, A)$.
(vi) If $(F, A)$ be a fuzzy soft group over $(X, A)$ and $f_j$ is a soft homomorphism such that $\varphi$ is onto and $\varphi$ is one-one, then $f_j[(F, A)]$ is an L-fuzzy soft group over $(Y, B)$.
(vii) If $(G_1, B) \geq (G_2, B)$ and $f_j$ is a soft homomorphism, then $f_j^{-1}[(G_1, B)] \geq f_j^{-1}[(G_2, B)]$. 

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(viii) If \((F, A) \xrightarrow{\Phi} (F, A)\) and \(f_j\) is a soft homomorphism such that \(f\) is onto and \(\varphi\) is one-one, then \(f_j \{[(F, A)] \xrightarrow{\Phi} f_j \{[(F, A)]\}\).

**Proof.** (i) Let \((F, A)\) be identity L-fuzzy soft group over \((X, A)\) and \(f_j\) is a soft homomorphism such that \(\varphi\) is onto. Then \(F(a) = 1_{[e_x]}, a \tilde{\rightarrow} A\). Since \(f\) is a homomorphism, it follows that \(f(e_x) = e_Y\) and for any \(y \tilde{\rightarrow} Y\),
\[
\{f[1_{[e_X]}](y) = \begin{cases} \cup_{x \in f^{-1}(y)}[1_{[e_X]}](x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{if otherwise} \end{cases} = \begin{cases} 1 & \text{if } y = e_Y \\ 0 & \text{if } y \neq e_Y \end{cases}
\]
Again since \(\varphi\) is onto, it follows that \(f^{-1}(b) = b \tilde{\rightarrow} B\) and hence for any \(b \tilde{\rightarrow} B\),
\[
f_{\varphi}[(F, A)](b) = \cup_{\alpha \in \varphi^{-1}(\beta)}f[F(\alpha)] = \cup_{\alpha \in \varphi^{-1}(\beta)}f[1_{[e_X]}] = 1_{[e_Y]}.
\]
Thus, \(f_j \{[(F, A)](b) = (\emptyset_{e_X}, A)\) and hence \(f_j \{[(F, A)]\) is identity L-fuzzy soft group over \((Y, B)\).

(ii) Let \((G, B)\) be identity L-fuzzy soft group over \((Y, B)\) and \(\varphi\) is a soft homomorphism such that \(\ker(f) = \{e_x\}\). Then \(G(b) = 1_{[e_y]}, b \tilde{\rightarrow} B\). Now for any \(a \in A\) and \(x \in X\),
\[
\{f^{-1}[G(\varphi(a))]\}(x) = [G(\varphi(a))](f(x)) = \begin{cases} [G(\varphi(a))](e_Y) = 1 & \text{if } x = e_X \\ 0 & \text{if } x \neq e_X \end{cases}
\]
and hence \(f_j^{-1}[(G,B)](a) = f^{-1}[Gj(a)] = 1_{[e_x]}\).

So, \(f_j^{-1}[(G,B)] = (\emptyset_{e_y}, A)\).

Therefore, \(f_j^{-1}[(G,B)]\) is identity L-fuzzy soft group over \((X, A)\).

(iii) Let \((F, A)\) be absolute L-fuzzy soft group over \((X, A)\) and \(f_j\) is a soft homomorphism such that \(f_j\) is onto.

Then \(f_j \{[(F, A)] = (\emptyset_{e_Y}, B) = U_{a}j^{-1}(b) f[F(a)] = U_{a}j^{-1}(b) f[1_{[e_X]}] = 1_{[e_Y]}\).

Since \(f\) is onto, it follows that \(f^{-1}(y) = y \tilde{\rightarrow} Y\) and for any \(y \tilde{\rightarrow} Y\),
\[
\{f[1_{[e_X]}](y) = \bigcup_{x \in f^{-1}(y)}[1_{[e_X]}](x) = 1.
\]
Again since \(\varphi\) is onto, it follows that \(f^{-1}(b) = b \tilde{\rightarrow} B\) and hence for any \(b \tilde{\rightarrow} B\), \(f_j \{[(F, A)](b) = U_{a}j^{-1}(b) f[F(a)] = U_{a}j^{-1}(b) f[1_{[e_X]}] = 1_{[e_Y]}\).

Thus, \(f_j \{[(F, A)] = (\emptyset_{e_Y}, B)\) and hence \(f_j \{[(F, A)]\) is absolute L-fuzzy soft group over \((Y, B)\).

(iv) Let \((G, B)\) be absolute L-fuzzy soft group over \((Y, B)\) and \(f_j\) is a soft mapping.

Then \(G(b) = 1_{[e_Y]}, b \tilde{\rightarrow} B\). Now for any \(a \in A\) and \(x \in X\),
\[
\{f^{-1}[Gj(a)]\}(x) = [Gj(a)](f(x)) = 1.
\]
Hence, \(f_j^{-1}[(G,B)](a) = f^{-1}[Gj(a)] = 1_{[e_X]}, a \tilde{\rightarrow} A\).

So, \(f_j^{-1}[(G,B)] = (\emptyset_{e_X}, A)\).

Therefore, \(f_j^{-1}[(G,B)]\) is absolute L-fuzzy soft group over \((X, A)\).
(v) Let \((G, B)\) be a L-fuzzy soft group over \((Y, B)\) and \(f_j\) is a soft homomorphism. Then \(G(\beta)\) is a L-fuzzy subgroup of \(Y, \forall \beta \in B\). Since \(f\) is a homomorphism, it follows that \(f^{-1}[G(b)]\) is an L-fuzzy subgroup of \(X, \forall \beta \in B\). Hence, \([f^{-1}_j[(G, B)](a)] = f^{-1}[G[j(a)]]\) is an L-fuzzy subgroup of \(X, \forall a \in A\). Therefore, \([f^{-1}_j[(G, B)](a)] = f^{-1}_j[(G, B)]\) is an L-fuzzy soft group over \((X, A)\).

(vi) Let \((F, A)\) be an L-fuzzy soft group over \((X, A)\) and \(f_j\) is a soft homomorphism such that \(f\) is onto and \(\phi\) is one-one. Then \(F(\alpha)\) is an L-fuzzy subgroup of \(Y, \forall \alpha \in A\). Again since \(\phi\) is one-one, it follows that either \(j^{-1}(b) = \emptyset\) or a singleton set. Now for any \(\beta \in B,\)
\[
f_{\phi}[(F, A)](\beta) = \begin{cases} \cup_{\alpha \in \phi^{-1}(\beta)} f[F(\alpha)] = f[F(\alpha)] & \text{if } \phi^{-1}(\beta) = \{\alpha\} \text{ (say)} \neq \emptyset \\
\emptyset & \text{otherwise}
\end{cases}
\]
Thus \([f_{\phi}[(F, A)](b)] = f_{\phi}[(F, A)]\) is an L-fuzzy subgroup of \(Y, \forall \beta \in B\). Therefore, \([f_{\phi}[(F, A)](a)] = f_{\phi}[(F, A)]\) is an L-fuzzy soft group over \((Y, B)\).

(vii) Let \((G_1, B), (G_2, B)\) be two L-fuzzy soft group over \((Y, B)\) such that \([f^{-1}(a)](x_i, x_j) = f[F(a)](x_i, x_j), x_i, x_j \in X\).
\((F_1, A) \subseteq (F_2, A)^n\) \(\forall \alpha \in A\).
\(f_j[(G, B)](a) = f^{-1}_j[(G, B)](a)\) \(\beta \in B\) \((G_1, B) \subseteq (G_2, B)\) and \(f_j\) is a soft homomorphism.
\(j_1(F_1, A) \subseteq (F_2, A) <_L F_2(a)\)
\(f_j^{-1}[G_1(b)](a) = [(f_j^{-1}[G_1(b)](a)]\) \(\forall \beta \in B\).
Thus \(f_j^{-1}[G_1(b)](a) = f^{-1}[G_1(j(a))][f^{-1}[G_2(j(a))]] = f_j^{-1}[G_2(b)](a) \beta \in B\).
Therefore, \(f_j^{-1}[G_1(b)](a) \subseteq f_j^{-1}[G_2(b)](a)\).

(viii) Let \((F_1, A), (F_2, A)\) be two L-fuzzy soft groups over \((X, A)\) such that \((F_1, A) \subseteq (F_2, A)\) and \(f_j\) is a soft homomorphism where \(f\) is onto and \(\phi\) is one-one. Then \(f_j [(F_1, A)], f_j [(F_2, A)]\) are L-fuzzy soft groups over \((Y, B)\) and \(f_j\) is a soft homomorphism such that \(f_j [(F_1, A)], f_j [(F_2, A)]\) are L-fuzzy soft groups over \((Y, B)\).
Hence, \( f_j \left[ (F_1, A) \right] \leq_{L} f_j \left[ (F_2, A) \right] \).

**Definition 4.8.** An L-fuzzy soft group \((F, A)\) over \((X, A)\) is said to be an L-fuzzy normal soft group over \((X, A)\) if \(F(a)\) is an L-fuzzy normal subgroup of \(X\), i.e., for any \( a \in A \), \([F(a)](x_1, x_2) = [F(a)](x_2, x_1)\), "\(x_1, x_2 \in X\).

**Proposition 4.9.** Let \((F_1, A)\), \((F_2, A)\) be two L-fuzzy normal soft group over \(X\). Then \((F_2, A) \leq_{L} (F_2, A)\) is an L-fuzzy normal soft group over \(X\).

**Proof.** Proof is straightforward.

**Proposition 4.10.** (i) Let \((F, A)\) be an L-fuzzy normal soft group over \(X\) and \(f_j\) be a soft homomorphism. If \(f\) is onto and \(j\) is one-one, then \(f_j \left[ (F, A) \right] = \left( f_j(F), B \right)\) is an L-fuzzy normal soft group over \(Y\).

(ii) Let \((G, B)\) be an L-fuzzy normal soft group over \(Y\) and \(f_j\) be a soft homomorphism. Then \(f_j^{-1} \left[ (G, B) \right] = \left( f_j^{-1}(G), A \right)\) is an L-fuzzy normal soft group over \(X\).

**Proof.** (i) Since \((F, A)\) is an L-fuzzy normal soft group over \(X\), it follows that \(F(a)\), "\(a \in A\) is an L-fuzzy normal subgroup of \(X\). Also since \(f\) is a homomorphism, \(f[F(a)]\) is an L-fuzzy normal subgroup of \(Y\). Again by part (vi) of Proposition 4.7, we have \(f_j \left[ (F, A) \right] \) is an L-fuzzy soft group over \(Y\) and for any \(b \in B\),

\[
\phi[ (F, A) ](b) = \left\{ \begin{array}{ll}
\bigcup_{\alpha \in \varphi^{-1}(b)} f[F(\alpha)] = f[F(\alpha)] & \text{if } \varphi^{-1}(b) = \{\alpha\} \text{ (say)} \neq \emptyset \\
0 & \text{otherwise}
\end{array} \right.
\]

Since, \(f[F(a)]\) and \(0_{(Y)}\) are both L-fuzzy normal subgroups of \(Y\), hence \(f_j \left[ (F, A) \right] (b)\) is an L-fuzzy normal subgroup of \(Y\), "\(b \in B\).

Therefore, \(f_j \left[ (F, A) \right] \) is an L-fuzzy normal soft group over \(Y\).

(ii) Since \((G, B)\) is an L-fuzzy normal soft group over \(Y\), it follows that \(G(b)\), "\(b \in B\) is an L-fuzzy normal subgroup of \(Y\). Also since \(f\) is a homomorphism, \(f^{-1}[G(b)]\) is an L-fuzzy normal subgroup of \(X\). Again by part (v) of Proposition 4.7, we have \(f_j^{-1} \left[ (G, B) \right] \) is an L-fuzzy soft group over \(X\) and for any \(a \in A\), \(f_j^{-1} \left[ (G, B) \right] (a) = f^{-1}[G(f^{-1}(a))]\) is an L-fuzzy normal subgroup of \(X\).

Therefore, \(f_j^{-1} \left[ (G, B) \right] \) is an L-fuzzy normal soft group over \(Y\).

**Definition 4.11.** Let \((F_1, A)\), \((F_2, A)\) be two L-fuzzy normal soft group over \(X\). Then \((F_1, A)\) is said to be L-fuzzy normal soft subgroup of \((F_2, A)\), denoted by \((F_1, A) \leq_{L} (F_2, A)\) if \(F_1(a) \leq_{L} F_2(a)\), "\(a \in A\).

**Proposition 4.12.** Let \((F_1, A)\), \((F_2, A)\) be two L-fuzzy normal soft group over \((X, A)\) and \((G_1, B)\), \((G_2, B)\) be two L-fuzzy normal soft group over \((Y, B)\). Also let \(f_j\) be a soft homomorphism.
(i) If \( (G_1, B) \cong (G_2, B) \), then \( f_j^{-1}[(G_1, B)] \cong (G_2, B) \).

(ii) If \( (F_1, A) \cong (F_2, A) \), \( f \) is onto and \( \varphi \) is one-one, then \( f_j[(F_1, A)] \cong (F_2, A) \).

Proof. Proofs follow from part (vii) and (viii) of Proposition 4.7.

Definition 4.13. Let \( N \) be a normal subgroup of \( X \) and \( \mu \) be an \( L \)-fuzzy subgroup of \( X \). Define
\[
\mu/N = \{ \mu(z) : z \in xN \} , \forall x \in X.
\]
Again \( \forall x, y \in X, \)
\[
(\mu/N)(xN)(yN) = \{ \mu(uv) : u \in xN, v \in yN \} = \{ [\mu(u) : u \in xN] \cup [\mu(v) : v \in yN] \}.
\]
Therefore \( \mu/N \) is an \( L \)-fuzzy subgroup of \( X/N \) and \( \mu/N \) is called \( L \)-fuzzy factor group of \( X/N \).

Definition 4.14. Let \( N \) be a normal subgroup of \( X \) and \( (F, A) \) be an \( L \)-fuzzy soft group over \( X \). Define a mapping \( f : A \rightarrow (F, A) \) by
\[
F/A = f(A) = \{ [F(a)](z) : z \in xN \} , \forall x \in X.
\]
Thus for each \( a \in A \), \( F/A \) is an \( L \)-fuzzy subgroup of \( X/N \) and hence \( F/A \) is an \( L \)-fuzzy soft factor group of \( (F, A) \) w.r.t. \( N \).

Definition 4.15. Let \( (F, A) \) be an \( L \)-fuzzy soft group over \( X \). Then \( F(a) \) is an \( L \)-fuzzy subgroup of \( X \) for each \( a \in A \). Now define a soft set \( (F^*, A) \), where \( F^*(a) = \{ x \in X : [F(a)](x) \neq 0 \} \), \( F^*(a) \) is an \( L \)-fuzzy subgroup of \( X \) and hence \( F^*(a) \) is an \( L \)-fuzzy soft group over \( X/N \) and is called the support of the \( L \)-fuzzy soft group \( (F, A) \).

Proposition 4.16. Let \( (F^*, A) \) be the support of the \( L \)-fuzzy soft group \( (F, A) \) over \( X \) and \( F^T(a) \) be the restriction of \( F(a) \) on \( F^*(a) \), \( a \in A \). Then \( F^T(a) \) is an \( L \)-fuzzy subgroup of \( F^*(a) \).

Proof. Let \( u, v \in F^*(a) \). Then
\[
[F^T(a)](uv^{-1}) = [F(a)](uv^{-1}) \geq [F(a)](u) \land [F(a)](v) = [F^T(a)](u) \land [F^T(a)](v).
\]
Therefore \( F^T(a) \) is an \( L \)-fuzzy subgroup of \( F^*(a) \).

Definition 4.17. Let \( (F, A) \) and \( (G, B) \) be two \( L \)-fuzzy soft groups over \( X, Y \) respectively. Then \( (F, A) \) is said to be \( f \)-fuzzy soft homomorphic to \( (G, B) \), denoted by \( (F, A) \sim (G, B) \), if for each \( (a, \beta) \in (A \times X, B) \), \( \exists \) an onto homomorphism \( f_{a, \beta} : F(a) \rightarrow G(b) \) such that \( f_{a, \beta}[F(a)] = G(b) \).

Proposition 4.18. Let \( (F, A) \) and \( (G, B) \) be two \( L \)-fuzzy soft groups over \( X, Y \) respectively such that \( (F, A) \) is \( f \)-fuzzy soft homomorphic to \( (G, B) \) and \( f_{a, \beta} \) be the corresponding onto homomorphism for \( (a, \beta) \in (A \times X) \). If \( K_{a, \beta} \) is the kernel of \( f_{a, \beta} \), then for each \( (a, \beta) \in (A \times X) \), \( \exists \) an isomorphism \( \hat{f}_{a, \beta} : F(a)/K_{a, \beta} \rightarrow G(b) \) and an \( L \)-fuzzy subgroup \( F(a) \) of \( F(a)/K_{a, \beta} \) such that \( \hat{f}_{a, \beta}[F(a)] = G(b) \).
Proof. Clearly \( K^{a,b} \) is a normal subgroup of \( F^*(a) \) and hence the mapping
\[
g^{a,b}_*: F^*(a) \cong F^*(a)/K^{a,b}
\]
defined by \( g^{a,b}_*(x) = xK^{a,b} \) is an onto homomorphism. Also the mapping
\[
\hat{f}^{a,b}_*: F^*(a)/K^{a,b} \cong G^*(b),
\]
defined by \( \hat{f}(xK^{a,b}) = f^{a,b}(x), \) \( x \in F^*(a) \) is an isomorphism. Now for each \( (a,b) \in A \times B \), we define \( F(a) \) by
\[
[F(a)](xK^{a,b}) = \bigvee \{ [F^*(a)](z) : z \in xK^{a,b} \}. 
\]
Again
\[
[F(\alpha)]((\xi K^{\alpha, \beta}) \eta K^{\alpha, \beta}) = [F(\alpha)]((\xi \eta K^{\alpha, \beta}) = \bigvee \{ [F^*(\alpha)](z) : z \in \xi \eta K^{\alpha, \beta} \} = \bigvee \{ [F^*(\alpha)](uv) : u \in \xi K^{\alpha, \beta}, \ v \in \eta K^{\alpha, \beta} \} \geq \bigvee \{ [F^*(\alpha)](u) : u \in \xi K^{\alpha, \beta} \} \wedge \bigvee \{ [F^*(\alpha)](v) : v \in \eta K^{\alpha, \beta} \} = [F(\alpha)]((\xi K^{\alpha, \beta}) \wedge [F(\alpha)]((\eta K^{\alpha, \beta})
\]
and \( [F(\alpha)]((\xi^{-1} K^{\alpha, \beta})^{-1}) = \bigvee \{ [F(\alpha)](z) : z \in \xi^{-1} K^{\alpha, \beta} \} = \bigvee \{ [F(\alpha)](w^{-1}) : w^{-1} \in \xi^{-1} K^{\alpha, \beta} \} = \bigvee \{ [F(\alpha)](w) : w \in \xi K^{\alpha, \beta} \} = [F(\alpha)]((\xi K^{\alpha, \beta}).
\]
Therefore, \( F(\alpha) \) is an l-fuzzy subgroup of \( F^*(a)/K^{a,b} \) and hence \( \hat{f}^{a,b}[F(a)] \) is an L-fuzzy subgroup of \( G^*(b) \).

Also for any \( y \in G^*(b) \), we have
\[
\{ \hat{f}^{\alpha, \beta}([F(\alpha)])(y) = \bigvee \{ [F(\alpha)]((\xi K^{\alpha, \beta}) : \hat{f}^{\alpha, \beta}(\xi K^{\alpha, \beta}) = y, \ \xi \in F^*(\alpha) \} = \bigvee \{ [F^T(\alpha)](z) : z \in \xi K^{\alpha, \beta} \} : \xi \in F^*(\alpha) \}
\]
\[
= \bigvee \{ [F^T(\alpha)](z) : \xi \in F^*(\alpha), \ f^{\alpha, \beta}(\xi) = y \} = [f^{\alpha, \beta}[F^T(\alpha)](y) = [G^T(\beta)](y).
\]
Therefore, \( \hat{f}^{a,b}[F(a)] = G^T(b) \).

5. Conclusion

In this paper, we have introduced L-fuzzy soft groups and studied some of its properties. Soft homomorphic image and pre image of L-fuzzy soft groups under soft mappings are given and fundamental homomorphism theorem is established in this setting. This study has a great importance for further research in generalization of the different structures in soft set theory viz. fuzzy soft topological group, fuzzy soft topological vector space etc.

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